CS 173 Lecture 5b: Set equalities

$A \cup B \qquad A \cap B \qquad \overline{A}$

DeMorgan: (i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ — propositions: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$ — $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Remarks: Sets $S = T$ iff $S \subseteq T$ & $T \subseteq S$.

since $S = T$ iff $\forall x,\ x \in S \leftrightarrow x \in T$

iff $\forall x, (x \in S \rightarrow x \in T) \wedge (x \in T \rightarrow x \in S)$

iff $S \subseteq T$ & $T \subseteq S$.

Proof. (i) First: show $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Suppose $x \in \overline{A \cup B}$. — Goal: $x \in \overline{A} \cap \overline{B}$

$x \in A \cup B$ iff $x \in A$ OR $x \in B$

This means $x \notin A \cup B$, i.e., $\neg(x \in A \vee x \in B)$. Apply DeMorgan's law for propositions, we get $x \notin A$ and $x \notin B$. $x \notin A$ means $x \in \overline{A}$ and $x \notin B$ means $x \in \overline{B}$, so $x \in \overline{A} \cap \overline{B}$

$x \in \overline{A}$ and $x \in \overline{B}$

Next, we show $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Suppose $x \in \overline{A} \cap \overline{B}$. Then $x \notin A$ and $x \notin B$. Applying DeMorgan's law, we get $\neg(x \in A \text{ or } x \in B)$, i.e., $x \notin A \cup B$, so $x \in \overline{A \cup B}$. □

Distributive Law

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ — propositions: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. — $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$\forall (x,y) \in A \times B,\ (x,y) \in C \times D$

Claim: Let A, B, C, D be sets.

If $A \subseteq C$, $B \subseteq D$, then $A \times B \subseteq C \times D$.

$A \times B = \{(x,y) : x \in A \ \& \ y \in B\}$.

Proof: Let A, B, C, D be sets such that $A \subseteq C$, $B \subseteq D$.

Proof: Let A, B, C, D be sets such that $A \subseteq C$, $B \subseteq D$.

Let $(x, y) \in A \times B$. By definition, $x \in A$, $y \in B$. Since $A \subseteq C$, $x \in C$.

Similarly, since $B \subseteq D$, $y \in D$.

This means $(x, y) \in C \times D$.

So $A \times B \subseteq C \times D$.

$\square$

proving set inclusions: unpack definition

$$A \subseteq B \text{ iff } \forall x,\ x \in A \rightarrow x \in B.$$

proving $p \rightarrow q$:

Suppose p. This means

Therefore q.

Suppose $x \in A$. This means

Therefore $x \in B$.

So $A \subseteq B$.