CS 173 Lecture 26: Proofs w/ Universal Quantifiers

- Direct Proof:
  - start with assumptions
  - make some derivations
  - end up at the desired conclusion

Claim: For all real numbers x & y, if $\underbrace{x \,\&\, y \text{ are rational}}_{\text{assumption}}$, then $\underbrace{x+y \text{ is rational.}}_{\text{conclusion}}$

x is rational if there exist integers a, b such that $b \neq 0$, and $x = \frac{a}{b}$.

Proof. Assume x & y are rational.
That means there are integers k, l, m, n such that $l \neq 0$ and $n \neq 0$, and $x = \frac{k}{l}$ and $y = \frac{m}{n}$.
Then $x+y = \frac{k}{l} + \frac{m}{n} = \frac{kn + lm}{ln}$.

$kn+lm$ is an integer, and so is ln.
Furthermore, since $l \neq 0$ and $n \neq 0$, ln is not zero.
So $x+y$ is the ratio of two integers $\frac{c}{d}$, ($c = kn + lm$, $d = ln$) such that d is not zero.
That means that $x+y$ is rational. □

Proof by contrapositive.

$$p \to q \equiv \neg q \to \neg p$$

"contrapositive"

Claim: For all integers a & b, if a+b is odd, then a is odd or b is odd.

an integer x is even if there exists an integer k such that $x = 2k$

an integer x is even if there exists an integer k such that $x=2k$,

x is odd if there exists an integer ℓ such that $x=2\ell+1$.

Proof by contrapositive: For all integers a & b, if $\underbrace{\text{a is even and b is even}}_{\text{assumption}}$, then $\underbrace{\text{a+b is even}}_{\text{conclusion}}$.

Assume a & b are even. Then there exist integers k & ℓ such that $a=2k$ and $b=2\ell$.

Then $a+b=2k+2\ell=2(k+\ell)$.

$k+\ell$ is an integer.

Thus $a+b$ is two times an integer, so $a+b$ is even. $\square$

Proof by Cases

Claim: For any real number x, if $|x-3|>10$, then $x^2>40$.

For real number y, $|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y<0. \end{cases}$

Proof. Assuming that $|\underbrace{x-3}|>10$.

is $x-3\geq 0$, or $x-3<0$?

First suppose $x-3\geq 0$. Then $|x-3|=x-3$, so the assumption $|x-3|>10$ means that $x-3>10$, i.e. $x>13$.

Then $x^2>13^2=169>40$.

Next, suppose $x-3<0$. Then $|x-3|=-(x-3)=3-x$. Then, $|x-3|>10$ means $3-x>10$, i.e., $\underline{-13>-x}$,* i.e., $x>13$. Once again, $x^2>13^2>40$. $\square$

$x > 13$. Once again, $x^2 > 13^2 > 40$. $\square$

During lecture, I did the calculation incorrectly. It should actually be:

$$-7 > x, \text{ i.e., } -x > 7 \text{ so } x^2 = (-x)^2 > 7^2 > 40.$$