CS 173 Review Session

Wednesday, 5 August, 2020 13:58

Prepared:
- Set Inclusion ✓
- One-to-One function ✓
- Relations ✓
- Tree Induction ✓

(These examples are all (supposed to be) slightly harder than the oral review portion)

Set Inclusion

(a) Show that $(A-B)\cup(B-A) \subseteq A\cup B$

(b) Give an example of sets A, B, C s.t.

$$A\cup B \not\subseteq (A-B)\cup(B-A).$$

In general: proving $X \subseteq Y$ means showing $\forall x,\ x\in X \to x\in Y$.

(a): Show that $\forall x,\ x\in(A-B)\cup(B-A) \to x\in A\cup B$.

Suppose $x\in(A-B)\cup(B-A)$

$\leftrightarrow x\in A-B \lor x\in B-A$

$\to (x\in A \land x\notin B) \lor (x\in B \land x\notin A)$

$p\land q \to p$ ∘ ∘ $\to (x\in A \lor x\in B) \land \underbrace{(x\in A \lor x\notin A)}_{T} \land \overbrace{(x\notin B \lor x\in B)}^{T}$

$\to x\in A \lor x\in B \quad \land (x\notin B \lor x\notin A).$

$\leftrightarrow x\in A\cup B$

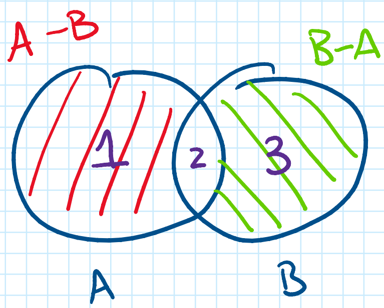
(b) $A=\{1,2,3\}$, $B=\{3,4,5\}$

$A\cup B=\{1,2,3,4,5\}$

$A-B=\{1,2\}$

$B-A=\{4,5\}$

$(A-B)\cup(B-A)=\{1,2,4,5\}$



Relations

$$S = P(\mathbb{N}) \times P(\mathbb{N})$$

$A,B,C,D \subseteq \mathbb{N}$

Define $\precsim$ on S s.t. $(A,B) \precsim (C,D)$

Define $\precsim$ on S s.t. $(A,B) \precsim (C,D)$ iff $A \subseteq C$ & $B \subseteq D$.

Prove: $\precsim$ is a partial order on S.

(1) $\precsim$ reflexive $\big(\forall (A,B) \in S,\ (A,B) \precsim (A,B) \big)$

Let $(A,B) \in S$. Since $A \subseteq A$ & $B \subseteq B$, $(A,B) \precsim (A,B)$.

(2) $\precsim$ antisymmetric $\left(\begin{array}{l} \text{if } (A,B) \precsim (C,D) \text{ \& } (C,D) \precsim (A,B) \\ \text{then } (A,B) = (C,D) \end{array} \right)$

Suppose $(A,B) \precsim (C,D)$ & $(C,D) \precsim (A,B)$.
So $A \subseteq C$ & $B \subseteq D$ & $C \subseteq A$ & $D \subseteq B$.
Then $A = C$ & $B = D$.
i.e. $(A,B) = (C,D)$.

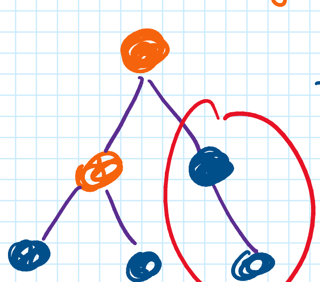
(3) $\precsim$ transitive $\left(\begin{array}{l} \text{if } (A,B) \precsim (C,D) \text{ \& } (C,D) \precsim (E,F) \\ \text{then } (A,B) \precsim (E,F) \end{array} \right)$

Suppose $(A,B) \precsim (C,D)$ & $(C,D) \precsim (E,F)$.
So $A \subseteq C$, $B \subseteq D$, $C \subseteq E$, $D \subseteq F$.
Then $A \subseteq E$ & $B \subseteq F$
i.e. $(A,B) \precsim (E,F)$.

Tree Induction

An **I**llini tree is a tree whose nodes are colored either orange or blue, where the leaves are all blue & root is orange.

Claim: in any Illini tree, there is a node that is orange and has a blue child.

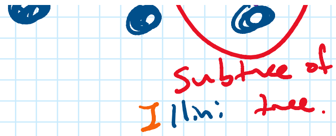


Subtree of

Proof: By induction on height h.

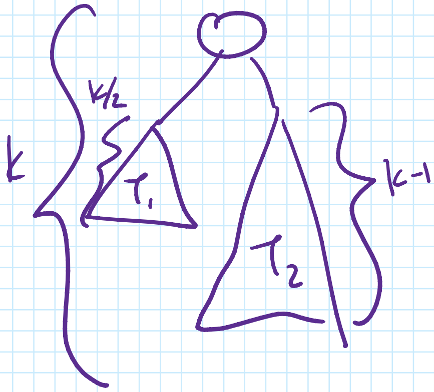
Base Case: $h=1$. If $h=0$, root is a leaf, so it would be both orange & blue, impossible.

root is orange
all children are leaves and therefore blue.



all children are leaves and therefore blue.

IH: For all Illini trees of height h, $0<h<k$, there is an orange node w/ a blue child.



IS: Let T be an Illini tree of height k.

Let $T_1 \dots T_\ell$ be the child subtrees of the root. All have height $< k$.

¡¡ Can only apply IH to Illini trees!!

Either

(1) $\exists\, T_i$ such that its root is orange (and leaves are blue)

$\rightarrow$ Illini

Apply IH to get an orange node w/ blue child.

(2) $\forall\, T_i$, T_i is not Illini

$\rightarrow$ root of T_i is blue. ¡¡no puedes aplicar IH!!

root is orange, all children are blue. $\square$

One-to-one : if $f(a)=f(b)$ then $a=b$ $\forall a,b$

Onto: $\forall b, \exists a$ s.t. $f(a)=b$.

Suppose $f:\mathbb{Z}\to\mathbb{Z}$ is one-to-one.

Let $g:\mathbb{Z}^2\to\mathbb{Z}^2$, $g(m,n)=(2f(m)+3f(n), -f(n))$ is also one-to-one.

Proof: Suppose $g(m,n)=g(a,b)$.

Want to show $(m,n)=(a,b)$.

$g(m,n)=g(a,b)$

$\rightarrow (2f(m)+3f(n), -f(n)) = (2f(a)+3f(b), -f(b))$

$\rightarrow 2f(m)+3f(n) = 2f(a)+3f(b)$ & $-f(n)=-f(b)$.

$\rightarrow 2f(m)+3f(n) = 2f(a)+3f(b)$ & $-f(n) = -f(b)$.

Since f is one-to-one & $f(n)=f(b)$,
$n=b$. Also, $3f(n)=3f(b)$, so
$2f(m) = 2f(a) \rightarrow f(m)=f(a)$.
So $m=a$.

So $(m,n)=(a,b)$.

$f: \mathbb{Z}^+ \rightarrow \mathbb{N}$

$f(1)=0$

$f(n) = 1+f(\lfloor n/2 \rfloor)$ for $n \geq 2$.

Prove: $f(n) \leq \log_2 n$ for all $n \geq 1$. (n is not necessarily a power of 2)

Proof: By Induction on n.

Base Case: $n=1$

$f(1)=0$ & $\log_2 1 = 0$

So $f(1) \leq \log_2 1$.

IH: Assume for $0 < k < n$ that $f(k) \leq \log_2 k$. $(n \geq 2)$.

IS: Want to show $f(n) \leq \log_2 n$.

$$
\begin{aligned}
f(n) &= 1+f(\lfloor n/2 \rfloor). \\
&\leq 1+\log_2 \lfloor n/2 \rfloor. \\
&= \log_2 2 + \log_2 \lfloor n/2 \rfloor \\
&= \log_2 (2 \cdot \lfloor n/2 \rfloor)
\end{aligned}
$$

if n is a power of 2, $\log_2 (2 \lfloor \frac{n}{2} \rfloor) = \log_2 n$.

$\lfloor n/2 \rfloor \leq n/2$.

$$
\begin{aligned}
\rightarrow \log_2 (2 \cdot \lfloor n/2 \rfloor) &\leq \log_2 (2 \cdot n/2). \\
&= \log_2 n.
\end{aligned}
$$