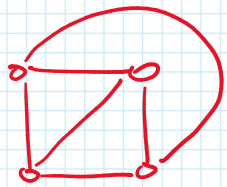
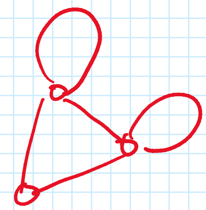
CS 173 Lecture 23a: Planar Graph & Curves

An undirected graph is <u>planar</u> if it can be drawn on $\mathbb{R}^2$ such that edges do not cross.

not necessarily simple
can have self-loops

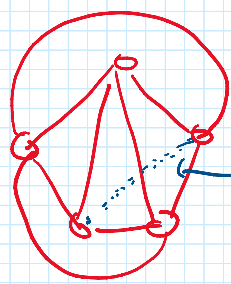


K_4



planar graph w/ loops

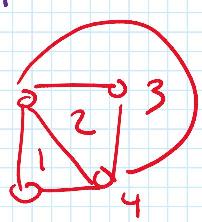
non-example: K_5 is not planar.

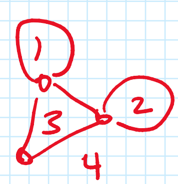


cannot draw one of the edges w/o crossing.

How to prove this? → Euler's Formula
↳ beginnings of topological graph theory
"topology" in general

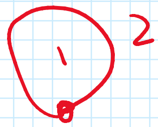
Before that: a drawing of a planar graph splits the plane into regions called "faces"





K_4 has 4 faces
3 "bounded" and one "outer" face

the idea that these curves split $\mathbb{R}^2$ into disjoint regions seems intuitive but was very difficult to prove.



The Classical Jordan Curve Theorem says that every simple closed curve in $\mathbf{R}^2$ divides the plane into two pieces, the "inside" and "outside" of the curve. Lest the theorem appear too obvious, try your intuition on the example shown in Figure 2-19.

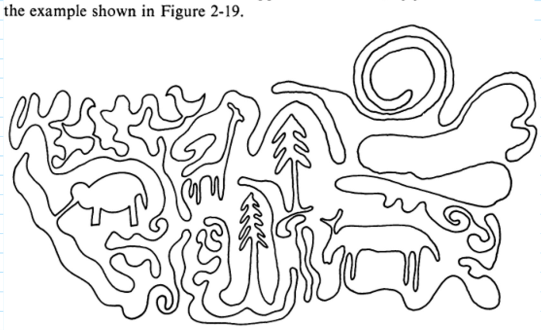


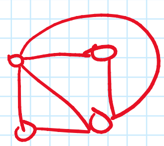
Figure 2-19

Guillemin-Pollack "Differential Topology"

given a planar graph drawing, it has V vertices, E edges, & F faces.

The most important thm in planar graphs
Forms the basis of most algorithms involving planar graphs

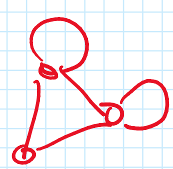
Thm (Euler's Formula) $V-E+F=2$ if G is connected.



K_4

$V=4$
$E=6$
$F=4$

$4-6+4=2.$ ✓



$V=3$
$E=5$
$F=4$

$3-5+4=2.$ ✓

Google "Geometry Junkyard Euler" → 20 different proofs.

Pf. By induction (on #edges)

Base Case: $E=0$.

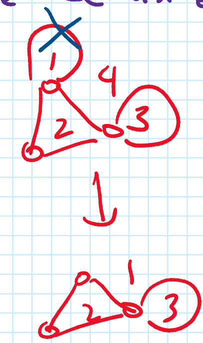
Graph is a single vertex

$V=1$, $E=0$, $F=1$ → $V-E+F=2$.

I.H. connected planar graph drawing w/ k edges has $V-k+F=2$ for $0 \le k < E$.

I.S. Let G be a planar graph w/ E edges,

Let e be an edge. if e is a loop, delete it

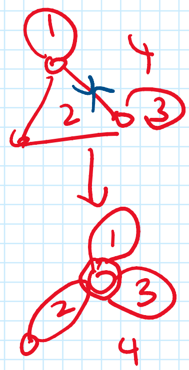


This gives a planar graph drawing G' w/ $E-1$ edges & $F-1$ faces.

$$IH \rightarrow V-(E-1)+(F-1)=2$$

$$\rightarrow V-E+F=2.$$

else, contract E. results in G' w/ $V-1$ vertices & $E-1$ edges.



$$IH \rightarrow (V-1)-(E-1)+F=2$$

$$\rightarrow V-E+F=2.$$

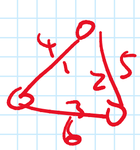
□

Cor: If G is simple then $E \leq 3V-6$.

Pf: Since G is simple every face is bounded by at least 3 edges.

each edge is bounded by 2 faces.

So $2E$ counts # of edge-face incidences



& $3F$ lower bounds # of edge-face incidences.

$$\rightarrow 2E \geq 3F$$

plugging into $V-E+F=2$

$$\rightarrow E \leq 3V-6.$$

□

Cor: avg deg of a planar graph $\leq 6-\frac{12}{V} < 6$. ← $\exists$ of a vtx w/ deg ≤ 5 fundamental of a planar graph $\le 6 - \frac{12}{V} < 6$. $\Leftarrow \exists$ of a vtx w/ deg ≤ 5 fundamental in planar graph theory & algs.

Cor: K_5 is not planar.

Pf. By contrapositive of cor:

If $E > 3V - 6$, then G is not a simple connected <u>planar</u> graph.

K_5: $V = 5$, $E = 10 > 9 = 3V - 6$.

Since K_5 is simple & connected, K_5 cannot be planar. $\square$