# CS 173 Lecture 22b: Uncountability

Consider $\{0,1\}^\infty$, the set of infinite binary sequences.

$$\underbrace{0110101011\ldots}_{\text{countably infinitely many digits}}$$

$$\prod_{i=0}^{\infty}\{0,1\}$$

$$(0,1,1,0,1,0,1,0,1,1,\ldots)$$

**Thm (Diagonalization)** $\{0,1\}^\infty$ is uncountable

Pf. $\{0,1\}^\infty$ is infinite, so we need to show there is no bijection $f : \mathbb{N} \to \{0,1\}^\infty$.

Let $f : \mathbb{N} \to \{0,1\}^\infty$.

We can notate $f(i)$ for $i \in \mathbb{N}$ as

$$f(i) = a_{i0}\ a_{i1}\ a_{i2}\ a_{i3}\ \cdots$$

$a_{ij} \in \{0,1\}$ for $j \in \mathbb{N}$.

| $n$ | $f(n)$ |
|---|---|
| 0 | $a_{00}\,a_{01}\,a_{02}\,a_{03}\cdots$ |
| 1 | $a_{10}\,a_{11}\,a_{12}\,a_{13}\cdots$ |
| 2 | $a_{20}\,a_{21}\,a_{22}\,a_{23}\cdots$ |
| 3 | $a_{30}\,a_{31}\,a_{32}\,a_{33}$ |

Let $b = b_0 b_1 b_2 \cdots$ where for each $i \in \mathbb{N}$,

$$b_i \neq a_{ii}.$$

$\to$ for all $i \in \mathbb{N}$, $b \neq f(i)$.

$\to$ $f$ is not onto

$\to$ $f$ is not bijective. $\square$

**Cor:** The following are uncountable:

(1) $P(\mathbb{N})$.

(2) $\{f : \mathbb{N} \to \{0,1\}\}$

(1) $P(\mathbb{N})$.

(2) $\{f : \mathbb{N} \to \{0,1\}\}$

(3) $(0,1)$

(4) $\mathbb{R}$.

{why diagonalization shows up in Real Analysis}

Pf Sketch. (1): $f : P(\mathbb{N}) \to \{0,1\}^\infty$

for $S \subseteq \mathbb{N}$, $f(S)$ is the string

$a_0 a_1 a_2 a_3 \cdots$ where $a_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{else} \end{cases}$

claim: $f$ is bijection.

(2) $g : P(\mathbb{N}) \to \{f : \mathbb{N} \to \{0,1\}\}$

for $S \subseteq \mathbb{N}$, $g(S) : \mathbb{N} \to \{0,1\}$,

$$g(S)(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{else.} \end{cases}$$

Claim: $g$ is bijective.

(3) $h : \{0,1\}^\infty \to (0,1)$

$$h(a_0 a_1 a_2 \cdots) = 0.b_0 b_1 b_2 b_3 \cdots$$

$$b_i = a_i + 3.$$

claim: $h$ is injective.

(4) $(0,1) \subseteq \mathbb{R}$. $\square$

{why diag shows up in CS courses}

Consequence

Thm: There exist problems for which there is no algorithm (there exist uncomputable problems)

Pf sketch: every computer program can be written as a finite binary string.

Claim: $|\mathbb{N}| = |\{0,1\}^*|$

$f(0) = \varepsilon$ $\quad f(3) = 00$

$f(1) = 0$ $\quad f(4) = 01$ $\quad \cdots$

$f(5) = 10$

$f(1) = 0$ $\qquad$ $f(4) = 01$ $\qquad$ ....

$f(2) = 1$ $\qquad$ $f(5) = 10$

$f(6) = 11$

Define for each $S \subseteq \mathbb{N}$, the problem

$Q_S =$ " given $i \in \mathbb{N}$, is $i \in S$? ".

$|\text{programs}| = \mathbb{N}$. $\qquad$ $|\{Q_S : S \subseteq \mathbb{N}\}| = |P(\mathbb{N})|$.

if $\quad g : \{\text{programs}\} \to \{\text{problems}\}$

no matter what $\quad \exists\, S \subseteq \mathbb{N}$, such that

$\nexists$ program $p$ s.t. $g(p) = Q_S$.

$\rightarrow$ there exists a problem $Q_S$ w/ no program solving $Q_S$. $\square$

$\longrightarrow$ CS 374, CS 475