CS 173 Lecture 22a: Countability

Recall: $|A| = |B|$ iff $\exists\, f: A \to B$, s.t. f is bijective

one-to-one & onto

Declare: this should be true even if A, B are infinite, e.g. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

$\longrightarrow$ different "infinities", might differ from expectations

Let $\mathbb{E} = \{2n : n \in \mathbb{N}\}$, the set of even natural numbers

Claim: $|\mathbb{N}| = |\mathbb{E}|$.

Pf: Let $f: \mathbb{N} \to \mathbb{E}$, $f(n) = 2n$.

f is one-to-one: $f(m) = f(n) \to 2m = 2n \to m = n$.

onto: given $2n \in \mathbb{E}$, $f(n) = 2n$. $\square$

Strange: 0, 1, 2, 3, 4, 5, 6, ...

"Intuitively", $\mathbb{E}$ has half the elements of $\mathbb{N}$.

Claim: $|\mathbb{N}| = |\mathbb{Z}|$

Intuition: $\mathbb{N}$: 0 1 2 3 4 5 6 7 8 ...

$\mathbb{Z}$: 0 −1 1 −2 2 −3 3 −4 4 ...

Pf: Let $f: \mathbb{N} \to \mathbb{Z}$, $f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -\frac{n+1}{2} & \text{if } n \text{ odd.} \end{cases}$

f is one-to-one: $f(m) = f(n) \to \begin{cases} \text{case } f(m)=f(n) \geq 0: \; m/2 = n/2 \to m = n \\ \text{case } f(m)=f(n) < 0: \; -\frac{m+1}{2} = -\frac{n+1}{2} \to m = n \end{cases}$

onto: given $z \in \mathbb{Z}$, $\begin{cases} z \geq 0, & f(2z) = z \\ z < 0, & f(-2z-1) = z. \end{cases}$ $\square$

Claim: $|\mathbb{N}| = |\mathbb{N}^2|$

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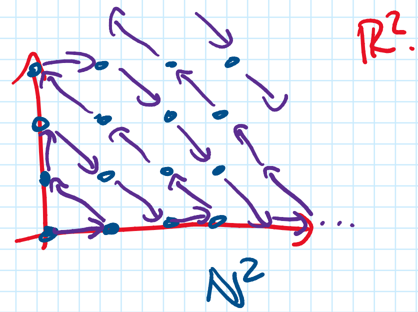
"Pf Sketch":



$\mathbb{N}$:	0	1	2	3	4	5	...
$\mathbb{N}^2$:	(0,0)	(1,0)	(0,1)	(0,2)	(1,1)	(2,0)	...

Refinement on this:

$|A| \le |B|$ iff $\exists\ f: A \to B$ s.t. f is one-to-one

2-way bounding

Thm (Cantor-Schroeder-Bernstein):

if $\exists\ f: A \to B$, $g: B \to A$, both one-to-one,

then $|A| = |B|$.

$|A| \le |B|$ & $|B| \le |A| \to |A| = |B|$.

Claim: $|\mathbb{N}| = |\mathbb{N}^2|$

Pf Sketch: Let $f: \mathbb{N} \to \mathbb{N}^2$, $f(n) = (0, n)$. obviously one-to-one

$g: \mathbb{N}^2 \to \mathbb{N}$ $\quad g(m,n) = 2^m 3^n$ one-to-one b/c FTA. $\square$

At this point: many sets w/ same card as $\mathbb{N}$.

Def: A set A is countable if $\exists$ one-to-one $f: A \to \mathbb{N}$

A is either finite or countably infinite.

($\exists$ bijection $f: A \to \mathbb{N}$).

Claim: for all finite $k \in \mathbb{N}$, $|\mathbb{N}| = |\mathbb{N}^k|$.

Pf Sketch: f (same as before)

$g : \mathbb{N}^k \to \mathbb{N}$ is $g(m_1, \ldots, m_k) = \prod_{i=1}^{k} p_i^{m_i}$

p_i is the i-th prime.

Thm: (1) countable union of countable sets is countable

(2) finite product of countable sets is countable

Pf: (1) Let $A = \bigcup_{i=0}^{\infty} A_i$. Since each A_i is countable,

exist one-to-one maps $f_i : A_i \to \mathbb{N}$.

Define $F : A \to \mathbb{N}^2$, via

$F(a) = (i, f_i(a))$ where

$i = \min\{j : a \in A_j\}$.

F is one-to-one: if

$(i, f_i(a)) = (j, f_j(b))$

$\to i = j$ & $f_i(a) = f_i(b)$

$\to a = b$.

(2) $B = \prod_{i=1}^{k} A_i$ $\quad G : B \to \mathbb{N}^k$

$G((a_1, \ldots, a_k)) = (f_1(a_1), \ldots, f_k(a_k))$.

one-to-one because

f_i's are. $\square$

Q: what about $\mathbb{R}$? what about countable product?

$\longrightarrow$ uncountable, see next part.