CS 173 Lecture 19c: Partitions

Given a base set A

Partition $\mathcal{C}$ a collection of subsets of A

- covers all of A
- no subsets are empty
- no overlap between subsets

$\{\{$Ian, Tanvi, Arravind, Noah$\}$, $\{$Patrick, Sahand, Alize, Dipro$\}\}$

is a partition of course staff, partitioned by section (A / B)

Another:

$\{\{$Ian, Patrick$\}$, $\{$Tanvi, Sahand$\}$, $\{$Arravind, Noah, Alize, Dipro$\}\}$

partitioned by role (instr, TA, CA).

Formal definition of partition of A:

$\mathcal{C} \subseteq P(A)$, s.t.

- $\displaystyle\bigcup_{S \in \mathcal{C}} S = A$ $\quad \{S_1 \cup S_2 \cup \ldots, S_i \in \mathcal{C}\}$
- $\emptyset \notin \mathcal{C}$
- $\forall S_i, S_j \in \mathcal{C},\ S_i \neq S_j \rightarrow S_i \cap S_j = \emptyset$.

$A = \{2, 5, 7, 8, 13, 21\}$

$p : A \rightarrow P(A)$ by

$p(n) = \{s \in A : \gcd(s, n) \neq 1\}$.

$M = \{p(n) : n \in A\}$

$p(n) = \{ s \in A : \gcd(s, n) \neq 1 \}.$

$$\mathcal{M} = \{ p(n) : n \in A \}$$

Q: Is $\mathcal{M}$ a partition?

What is $\mathcal{M}$?

Some examples of sets in $\mathcal{M}$.

$p(13) = \{13\}$ $\quad$ $p(2) = \{2, 8\} = p(8)$

$$p(7) = \{7, 21\} = p(21).$$

$\forall n \in A$, is $n \in S$ for some $S \in \mathcal{M}$?

Yes, $n \in p(n)$.

for all $S \in \mathcal{M}$, is $S \neq \emptyset$?

Yes, if $S \in \mathcal{M}$, $S = p(n)$ for some n, and $n \in p(n)$.

if $S, T \in \mathcal{M}$, are S & T disjoint?

Yes: one way to verify check each set in $\mathcal{M}$ directly

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$A = \{2, 3, 4, 5, 10, 12\}$

$f : A \to \mathcal{P}(A)$, $\quad f(n) = \{ p \in A : p \mid n \}$

$$\mathcal{S} = \{ f(n) : n \in A \}$$

Is $\mathcal{S}$ a partition?

$f(12) = \{2, 3, 4, 12\}.$

$f(10) = \{2, 5, 10\}.$

$f(10) \neq f(12)$ but $f(10) \cap f(12) = \{2\}.$

$f(10) \neq f(12)$ but $f(10) \cap f(12) = \{2\}$.

NO! not a partition.

Partitions vs Equivalence Relations.

Given a Partition $\mathcal{C}$, define a relation on A

$x \sim y$ iff x, y are in the same part S of the partition $\mathcal{C}$.

Claim: $\sim$ is an eq. rel.

$S \in \mathcal{C}$

Pf: (1): reflexivity: $x \sim x$ since $x \in S$ iff $x \in S$

(2): Symmetry: $x \sim y \to x, y \in S$
$\to y, x \in S$
$\to y \sim x$.

(3) transitivity: $x \sim y \wedge y \sim z$
$\to x, y \in S \wedge y, z \in S$
$\to x, y, z \in S$
$\to x, z \in S$
$\to x \sim z$. $\square$

Let $\sim$ be a relation of A. Then the collection of equivalence classes
$\mathcal{C} = \{[x] : x \in A\}$ is a partition.

Pf. (1): covering:
$\forall x \in A, \quad x \in [x]$.

(2): nonempty:
$\forall S \in \mathcal{C}, \quad S = [x]$ for some x. $x \in [x]$.

(3): nonoverlapping:

Let $[x]$, $[y]$ be equivalence classes,

want to prove: $[x] \neq [y] \rightarrow [x] \cap [y] = \emptyset$.

Contrapositive: $[x] \cap [y] \neq \emptyset \rightarrow [x] = [y]$.

Let $z \in [x] \cap [y]$, then $x \sim z$ & $z \sim y$.

$[x] = \{z : z \sim x\}$

By transitivity, $x \sim y$. By symmetry, $y \sim x$.

By transitivity, $\forall w \in [x]$, $w \sim x$, so $w \sim y \rightarrow w \in [y]$

$[x] \subseteq [y]$.

Symmetrically, $\forall w \in [y]$, $w \sim y$, so $w \sim x$, $\rightarrow w \in [x]$

$[y] \subseteq [x]$

$\rightarrow [x] = [y]$.

$\square$

Actually properties of eq rel are chosen so that $\{$eq classes$\}$ is a partition & vice versa.