CS 173 Lecture 19b: Counting Subsets / Stars

$P(S) = \{T : T \subseteq S\}$ $\qquad |P(S)| = 2^{|S|}$

Fix $n = |S|$ $\qquad$ Q: How many subsets of S of cardinality k?

$\rightarrow \sum_{k=0}^{n}$ (# subsets of S of size k) $= |P(S)| = 2^{|S|}$

Recall: to get $|P(S)| = 2^{|S|}$, we asked how to define a subset T:

for each element $x \in S$, choose if $x \in T$.

$\rightarrow 2^n$ choices.

subsets of card k: how to define a subset T w/ $|T| = k$: from n elements of S, we choose k to be in T.

"Binomial coefficient"

Th: 3 is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$P(\{0,1\}) = \{ \underbrace{\phi,}_{\binom{2}{0}=1} \underbrace{\{0\}, \{1\},}_{\binom{2}{1}=2} \underbrace{\{0,1\}}_{\binom{2}{2}=1.} \}.$

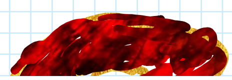
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

[special case of Binomial thm]

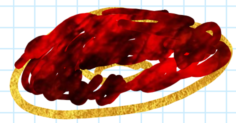
"Picking from n choices w/o repeats"

Q: what about w/ repeats?

Donut shop



Donut shop



Today: donut shop offering 3 kinds of donuts

Want to buy 10 donuts.

How many ways to buy 10 donuts of 3 different kinds?

equivalent to asking for $x_1, x_2, x_3 \in \mathbb{N}$,

such that $\underbrace{x_1}_{\substack{\#\text{ donuts}\\ \text{of type 1}}} + \underbrace{x_2}_{\substack{\#\text{ donuts}\\ \text{of type 2}}} + \underbrace{x_3}_{\substack{\#\text{ donuts}\\ \text{of type 3}}} = 10.$ ← total # of donuts.

In general: pick k items (w/ repeats) from n kinds of items

equivalent to $x_1, \dots x_n \in \mathbb{N}$,

s.t. $\sum_{i=1}^{n} x_i = k$.

Goal: count # of solutions to $\sum_{i=1}^{n} x_i = k$, $x_i \dots x_n \in \mathbb{N}$.

Technique: Stars & bars diagram

$$\underbrace{*\ *\ *}_{x_1=3}\ \Big|\ \underbrace{*\ *}_{x_2=2}\ \Big|\ \underbrace{*\ *\ *\ *\ *}_{x_3=5}$$

Draw k stars, & $n-1$ bars.

x_1 = # of stars before bar 1

x_2 = # stars between bars 1 & 2

⋮

⋮

x_n = # stars after last bar.

To count: $k+n-1$ objects (stars or bars)
and then $n-1$ of them need to be bars
(the rest are stars).

A: $\underbrace{\binom{k+n-1}{n-1}}_{\text{hard to memorize wrt } k \text{ \& } n.}$.

Stars & bars diagram: easy to reproduce
deriving this formula from stars & bars diagram
is easier (probably) than memorization