CS 173 Lecture 17c: Analyzing Recursion

$\overbrace{A[l..u]}^{\text{part of an array starting at index } l \text{ \& ending at index } u}$

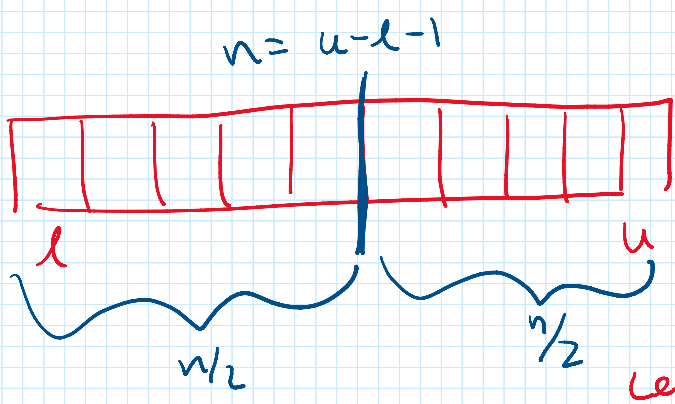
MergeSort(A[l..u]):
```
1.  if (u-l ≥ 1):
2.       m = ⌊(u-l)/2⌋
3.       MergeSort(A[l..m])
4.       MergeSort(A[m+1..u])
5.       Merge(A[l..u], m)
```

Merge(A[l..u], m):

(omitted)

recursive version in BBfTCS (CS173 text)

iterative version in algorithms.wtf (CS 374 text).



if line 1 returns true:

recursively sorts first half & second half

"merges" afterwards

Merge takes $\leq dn$ time for some $d \in \mathbb{R}$.

Let $f(n)$ be running time of MergeSort on input of size n

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Line 1: always executes & takes c_1 time.

Lines 2-4 execute if $n > 1$.

Line 2: takes c_2 time

Line 3: $f(\lfloor n/2 \rfloor)$ time.

Line 4: $f(\lceil n/2 \rceil)$ time

Line 5: $\leq dn$ time.

$$f(1) = c_1$$

$$f(n) \leq c_1 + c_2 + f(\lfloor n/2 \rfloor) + f(\lceil n/2 \rceil) + dn$$

assume n is a power of 2:

know from Lecture on Recursion trees:

$$f(n) \leq k_1 n(\log_2 n - 1) + k_2 n + k_3$$

Asymptotic analysis:

$$f(n) = O(n \log n)$$

unbalanced

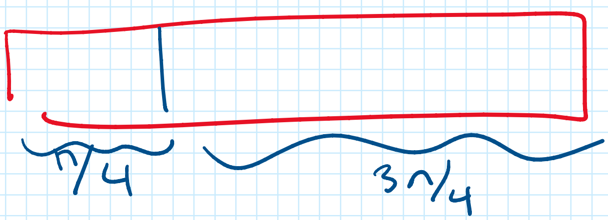
UMergeSort(A[l..u]):

```
if (u-l ≥ 1):
    m = ⌊(u-l)/4⌋
    UMergeSort(A[l..m])
    UMergeSort(A[m+1..u])
    Merge(A[l..u], m)
```

$g(1) = c_1$

$g(n) \le c_1 + c_2 + \underbrace{g(\lfloor n/4 \rfloor) + g(\lceil 3n/4 \rceil)}_{}$ $+ dn$

unbalanced? floors/ceilings?



floors & ceilings: ignore them

justification: replace $g(n): \mathbb{Z} \to \mathbb{R}$

w/ upper bound $h(x): \mathbb{R} \to \mathbb{R}$.

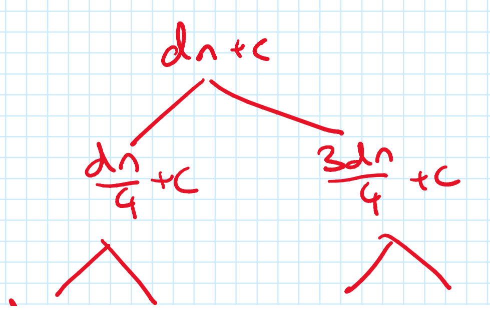
if done right,

$g(n) = O(h(n))$. kind of a lie

full details: algorithms.wtf.

unbalanced recursion:

$$g(n) = g\left(\frac{n}{4}\right) + g\left(\frac{3n}{4}\right) + dn + c$$



total extra work?

$\le dn + c$

$\le dn + 2c$

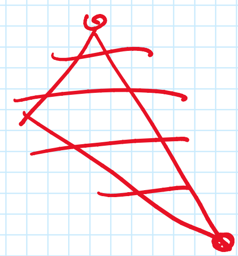
$\frac{dn}{16}+c \quad \frac{3dn}{16}+c \qquad \frac{3dn}{16}+c \quad \frac{9dn}{16}+c \qquad \le dn+3c$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\le dn+\ell c$

at some level ℓ, $\frac{dn}{4^\ell} \le 1$.

height $> \ell$ since $\frac{3^\ell dn}{4^\ell} > 1$.



$\le dn+\ell c$

at $\ell = h$, base case c,

$$\left(\tfrac{3}{4}\right)^h n \le 1$$
$$\iff n \le \left(\tfrac{4}{3}\right)^h$$
$$\iff \log_{\frac{4}{3}} n \le h$$

$$g(n) \le \left(\sum_{\ell=0}^{h-1} dn+\ell c\right) + c,$$
$$\le \left(\sum_{\ell=0}^{\log_{\frac{4}{3}} n} dn+\ell c\right)+c,$$
$$= O(n \log n).$$