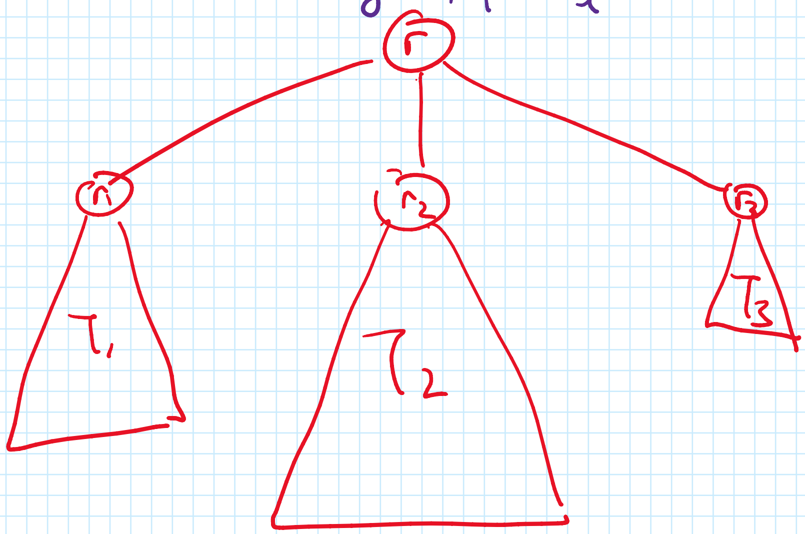
CS 173 Lecture 13b: Recursive Definition of & Induction on Trees

Recursive Def of Trees:

Base Case: A single vertex is a tree.
↑ root.

Recursive Case: Let $T_1 \dots T_\ell$ be trees (that don't share vertices)
w/ roots $r_1 \dots r_\ell$, resp.

A new tree can be constructed by taking a new vertex r ← root
and making $r_1 \dots r_\ell$ the children of r.



Induction on Trees: Induction on $h \geq 0$ (height of Tree)

Base Case $h = 0$. a single vertex.

I.H.: Assume for $0 \leq k < h$ that
$P(k)$ holds for all trees of height k.

I.S. Want to show $P(h)$ holds for all trees of height $h > 0$.
Let T be an arbitrary Tree of height $h > 0$.
w/ root r.

Let $T_1 \dots T_\ell$ be the subtrees rooted at the children $r_1 \dots r_\ell$ of r.

The heights of $T_1 \dots T_\ell$ are all less than h.

The I.H. implies that P holds for $T_1 \dots T_\ell$

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⋮ ∞ {steps go here}

Therefore P holds for T.
Since T was an arbitrary tree of height h,
$P(h)$ holds for all trees of height h.

"Def" A galactic tree is a full binary tree w/ natural number labels on the nodes, such that the leaves have labels 5, 10 or 15. & a node whose children have labels x & y has label xy.

Claim: A galactic tree's root has label divisible by 5.

Proof: Let $P(h)$: "The root of a galactic tree of height h has label divisible by 5".

We prove $P(h)$ by induction.

Base Case: $h=0$. the root is a leaf, so its label is 5, 10, or 15. all are divisible by 5.

IH: For $0 \le k < h$, assume $P(k)$ holds for all galactic trees of height k.

IS: Let T be a galactic tree of height $h > 0$ w/ root r. r has two children u & v. Let T_u & T_v be the subtrees rooted at u & v, resp.

The heights of T_u & T_v are less than h, and T_u & T_v are galactic

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So IH implies that the labels of
u & v are divisible by 5. i.e.
if p & q are the labels, then $5 \mid p$ & $5 \mid q$.
The label of r is pq, and so $5 \mid pq$.
So $P(h)$ holds for all galactz trees
of height h. $\square$

Induction on trees: always split the tree T
into root + subtrees.
Apply I.H. to subtrees,
⋮
prove P holds on T.

HW: Look at the proof in §11.8
and understand how it follows the outline.