CS 173 Lecture 12c: Proving Closed Forms

Induction $\xrightarrow{\text{flip}}$ Recursion

Q_n: $E_n = 2E_{n-1} + 2^{n-1}$, $E_0 = 0$.

Claim: $E_n = n2^{n-1}$ for $n \geq 0$

Proof. By Induction.

Base Case: $E_0 = 0$.

I.H.: For $0 \leq k < n$, $E_k = k2^{k-1}$

I.S.: $E_n = 2E_{n-1} + 2^{n-1}$

$= 2(n-1)2^{n-2} + 2^{n-1}$

$= (n-1)2^{n-1} + 2^{n-1}$

$= n2^n$ $\square$

$T(n) = 2T(\frac{n}{2}) + n$, $T(2) = 1$, $n \geq 2$, n power of 2.

Claim: For $n \geq 2$ power of 2, $T(n) = n[(\log_2 n) - \frac{1}{2}]$

Proof. By induction.

Base Case: $n = 2$, $T(n) = 1$

I.H. for $2 \leq k < n$, k power of 2,

$T(k) = k[(\log_2 k) - \frac{1}{2}]$.

I.S. $T(n) = 2T(\frac{n}{2}) + n$

$= 2\frac{n}{2}[(\log_2 \frac{n}{2}) - \frac{1}{2}] + n$

$= n[(\log_2 n - \log_2 2) - \frac{1}{2}] + n$

$= n[(\log_2 n) - \frac{1}{2}] - n + n$

$= n[(\log_2 n) - \frac{1}{2}]$. $\square$

Odd Fibonacci: $F(0)=0, \quad F(1)=1$

$F(n) = F(n-1) + F(n-2)$.

Claim: $F(3n+1)$ is odd for all $n \geq 0$.

Proof: By Induction.

B.C. $n=0$: $F(1)=1$ which is odd.

I.H. Assume for $0 \leq k < n$ $F(3k+1)$ is odd.

I.S. $F(3n+1) = F(3n) + F(3n-1)$.

$$= F(3n-1) + F(3n-2) + F(3n-1)$$

$$= 2F(3n-1) + \underbrace{F(3(n-1)+1)}_{\text{odd.}}$$

same

$$= 2F(3n-1) + (2l+1) \quad \text{for some } l \in \mathbb{Z}.$$

$$= 2(F(3n-1)+l) + 1.$$

which is odd.

$\square$