# CS 173 Lecture 126: Unrolling

Closed form

$Q_n$ $V_n = \#$ vertices, $V_n = 2V_{n-1}$, $V_0 = 1$ $\rightarrow$ $V_n = 2^n$

$E_n = 2E_{n-1} + 2^{n-1}$, $E_0 = 0$ $\rightarrow$ $E_n = ?$

$\#$ edges

## Unrolling

Scratchwork, not a proof

technique for <u>guessing</u> closed form of a recursively defined function

1. "unroll" a few times

   (plug the definition into itself)

   1. $E_n = 2E_{n-1} + 2^{n-1}$

      ( $E_{n-1} = 2E_{(n-1)-1} + 2^{(n-1)-1} = 2E_{n-2} + 2^{n-2}$ )

   2. $= 2(2E_{n-2} + 2^{n-2}) + 2^{n-1} = 2^2 E_{n-2} + 2 \cdot 2^{n-1}$

      ( $E_{n-2} = 2E_{n-3} + 2^{n-3}$ )

   3. $= 2^2(2E_{n-3} + 2^{n-3}) + 2 \cdot 2^{n-1} = 2^3 E_{n-3} + 3 \cdot 2^{n-1}$

2. <u>Predict</u> the general case

   k. $= 2^k E_{n-k} + k2^{n-1}$

3. Find k to reach base case

   when $k = n$, $E_{n-k} = E_0$

   Base case: $E_0 = 0$

4. Substitute base case value (and simplify)

   $k = n$: $E_n = 2^n E_0 + n2^{n-1}$

   $= n2^{n-1}$

5. Sanity check.

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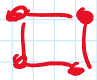
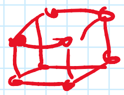
$Q_0$ :  $E_0 = 0$

$Q_1$ :  $E_1 = 1 = 1 \cdot 2^0$ ✓

$Q_2$ :  $E_2 = 4 = 2 \cdot 2^1$ ✓

$Q_3$  $E_3 = 12 = 3 \cdot 2^2$ ✓

Another example : for $n$ powers of 2, $n \geq 2$.

$$T(n) = 2T\left(\frac{n}{2}\right) + n, \quad T(2) = 1$$

1. "unroll" a few times

1. $T(n) = 2T\left(\frac{n}{2}\right) + n$

$$\left\{ T\left(\frac{n}{2}\right) = 2T\left(\frac{\left(\frac{n}{2}\right)}{2}\right) + \frac{n}{2} = 2T\left(\frac{n}{4}\right) + \frac{n}{2} \right.$$

2. $= 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n = 2^2 T\left(\frac{n}{2^2}\right) + 2n$

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \frac{n}{4}$$

3. $= 2^2\left(2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + 2n = 2^3 T\left(\frac{n}{2^3}\right) + 3n$

2. Predict

k. $= 2^k \, T\left(\underline{\frac{n}{2^k}}\right) + kn$

3. Find k meeting base case

$$\frac{n}{2^k} = 2 \iff n = 2^{k+1} \iff k = (\log_2 n) - 1$$

4. Plug in base case & simplify.

$$T(n) = 2^k \, T\left(\frac{n}{2^k}\right) + kn, \quad \text{when } k = (\log_2 n) - 1,$$

$$T(n) = \underbrace{2^{(\log_2 n) - 1}}_{2^{\log_2 n} \cdot 2^{-1}} T(2) + ((\log_2 n) - 1)\, n.$$

$$= \frac{n}{2}\cdot 1 + n\log_2 n - n$$

$$= n\left[(\log_2 n) - \frac{1}{2}\right].$$

5. Sanity Check

$T(2) = 1 \qquad 2(\log_2 2 - \frac{1}{2}) = 2(1-\frac{1}{2}) = 1$ ✓

$T(4) = 2T(2) + 4$
$= 2\cdot 1 + 4 = 6 \qquad 4(\log_2 4 - \frac{1}{2}) = 4(2-\frac{1}{2}) = 6$ ✓

$T(8) = 2T(4) + 8$
$= 2\cdot 6 + 8 = 20 \qquad 8(\log_2 8 - \frac{1}{2}) = 8(3-\frac{1}{2}) = 20$ ✓