CS 173 Lecture 10b: Why does induction work?

P.o.I: $\left[P(a) \wedge (\forall n \geq a, P(a) \wedge P(a+1) \wedge \dots \wedge P(n-1) \rightarrow P(n)) \right]$
$\rightarrow \forall n \geq a, P(n)$

Well Ordering Principle: Every non empty subset S of $\mathbb{Z}$ w/ a lower bound, then S has a smallest element.

$\exists a \in \mathbb{Z}$ s.t. $\forall s \in S, a \leq s$

Theorem: W.O.P. $\rightarrow$ P.o.I.

Pf (Sketch): By contrapositive:

$P(a)$ and $\forall n \geq a, P(a) \wedge \dots \wedge P(n-1) \rightarrow P(n)$

but $\exists n \geq a \ \neg P(n)$

Let $S = \{ n \geq a : \neg P(n) \}$.

$\{$ a is a lower bound of S

Let $k \in S$, Then

$\neg P(k) \rightarrow \neg P(a) \vee \dots \vee \neg P(k-1)$,

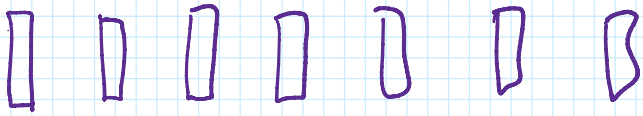
so $\exists \ell < k$ s.t. $\neg P(\ell)$.

i.e. $\ell \in S$ such that $\ell < k$.

Therefore S has no smallest element $\square$

Intuition for P.o.I:

1) Dominos.



If dominos are set up so that if a domino falls, it knocks down dominos next to it.

$P(a)$ $\wedge$ (knocking down the first domino

it knocks down dominos next to it.

$P(a)$ $\Leftrightarrow$ (knocking down the first domino.)

$P(a) \wedge P(a+1) \wedge \dots \wedge P(n-1) \rightarrow P(n)$

$\Leftrightarrow$ (every domino can be knocked over by some preceding domino.)

2) Recursion Fairy.

Suppose you want to know that $P(3000)$ is true.

and know that $P(a)$ is true

and $P(a) \wedge \dots \wedge P(n-1) \rightarrow P(n)$ $\forall n \geq a$.

$P(n)$ is true if all of these are true.

So to prove $P(3000)$, first prove that

$P(a), P(a+1), \dots P(2999)$ are true... right?

No! If you want $P(3000)$, the magic recursion fairy fills in proofs of $P(a), P(a+1) \dots P(2999)$, using,

~~$P(a) \rightarrow P(a+1)$.~~

~~$P(a) \wedge P(a+1) \rightarrow P(a+2)$.~~

~~$P(a) \wedge P(a+1) \wedge P(a+2) \rightarrow P(a+3)$~~

Never think about this.

Let  do this for you.

Since you explained how to do it for general n.

for general n.

Then $P(0) \dots P(2999)$ are true, b/c

fairy.

You already showed $P(0) \wedge \dots \wedge P(2998) \to P(3000)$

So, $P(3000)$ is true.