Tree Induction

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Learning Objective

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- Prove a claim about trees using induction.

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- I.H. for a particular value of the variable covers an entire *family* of trees
- Always divide tree up at the top (root plus subtrees of its children)

A Claim about Full Binary Trees

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Inductive step: Let $k > 0$ be an arbitrary natural number.

Fearsome Trees

Definition

A ***fearsome*** tree is a binary tree with each node labeled with a positive integer, such that:

1. If v is a leaf node, then v is labeled with 4 or 12.
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