Induction, Episode V: The Recursion Fairy Strikes Back

Part b: Picking Base Cases

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Learning Objectives

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- Explain why proofs with too few base cases break.

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- **A**: Yes! Watch out for this.

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Hence by induction, every natural number is irrational. □

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We then have

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