Induction, Episode IV: A New Proof Technique

Part b: A Full, Slow Example

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- Use induction to prove a formula is correct for all integers starting at some n_0.

Our First Inductive Proof Together

Claim

For every natural number n, $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

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We prove the claim by induction on n.
Let $P(n)$ be the statement $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
Base case: We need to show $P(0)$ is true.

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Base case: We need to show $P(0)$ is true.
Inductive step: Let $k > 0$ be arbitrary. Suppose that $P(n)$ is true for $n = 0, 1, \ldots, k-1$. (This is called the ***inductive hypothesis***, or I.H. for short.) We need to show that $P(k)$ is true.

□

Full Proof

Proof.

We prove the claim by induction on n. $P(n) := \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Base case: When $n = 0$, $\sum_{i=0}^{0} i^2 = 0^2 = \frac{0(0+1)(2\cdot 0+1)}{6}$, so $P(0)$ is true.

Inductive step: Let $k > 0$ be arbitrary. Suppose that $P(n)$ is true for $n = 0$ through $n = k - 1$. We have

$$\begin{aligned}
\sum_{i=0}^{k} i^2 &= k^2 + \sum_{i=0}^{k-1} i^2 \\
&= k^2 + \frac{(k-1)(k-1+1)(2(k-1)+1)}{6} && \text{(by the I.H.)} \\
&= \frac{2k^3 + 3k^2 + k}{6} && \text{(by algebra)} \\
&= \frac{k(k+1)(2k+1)}{6} && \text{(by factoring).}
\end{aligned}$$

Hence $P(k)$ is true.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Reminder: Parts of Inductive Proof

$$\left[S(n_0) \wedge \left[\forall k > n_0 \left[\bigwedge_{n=n_0}^{k-1} S(n) \rightarrow S(k)\right]\right]\right] \rightarrow \forall n \geq n_0 \; S(n).$$

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