Number Theory: Prime Numbers

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- Remember that there are infinitely many prime numbers.

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- Essential in cryptography
- Give us handy prime factorizations (short way of representing large integers)

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Exercise

Rewrite the definition of prime in logical notation.
Let P denote the set of prime numbers.

Special Cases

What about 0 and 1?

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Neither prime nor composite

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- $173 = 173$

Fundamental Theorem of Arithmetic

Theorem (Euclid)

*Every integer greater than one has a **unique*** prime factorization.*

* Ignoring the order in which we write the factors

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Proof sketch

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$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 1 = 510,511 = 19 \cdot 97 \cdot 277$

Recap: Learning Objectives

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