Combinatorial Proofs

Ian Ludden

Learning Objective

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- Prove combinatorial identities by counting the same quantity in two ways.

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Definition

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We've seen this before...

For all $n, k \in \mathbb{N}$, prove   Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Suppose we want to pick k elements from a set S with $|S| = n+1$, $S = \{s_1, s_2, \ldots, s_{n+1}\}$.

With s_1: $\binom{n}{k-1}$

W/o s_1: $\binom{n}{k}$

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General strategy to prove $A = B$:

1. Invent a counting problem you can solve in two ways.
2. Show that one answer to the counting problem is A.
3. Show that another answer is B.

BYO Word Problem

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Sum of binomial coefficients

Prove:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Know: 2^n is $|\mathcal{P}(S)|$ when $|S| = n$.

LHS: Condition on size of the subset.

\# subsets of S of size k: $\binom{n}{k}$

Possible sizes are $k = 0, 1, 2, \ldots, n$

So LHS counts $|\mathcal{P}(S)|$.

Sum of binomial coefficients

Prove:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Ascending/descending products

Prove:

$$\sum_{k=2}^{n+1} (k-1)(n+2-k).$$

$$1 \cdot n + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n \cdot 1 = \binom{n+2}{3}.$$

$k=2$ $k=3$ $k=n+1$

Problem: How many subsets of size 3 from $\{1,2,3,\ldots,n+2\}$.

RHS clearly counts this. Fix $b=k$.

LHS: Say we pick a,b,c, with $a<b<c$. There are $k-1$ opt. for a

Condition on b; can be from 2 to $n+1$. $(n+2)-k = n+2-k$ for c

More Examples

More Examples

$\binom{n}{k} = \binom{n}{n-k}$. $\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \dots + \binom{n}{n}\cdot\binom{n}{0}$

Sum of *squares* of binomial coefficients

Prove:

$\binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \dots + \binom{n}{n}\binom{n}{n}$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

RHS: Choose n from set of size $2n$.

Prof. Snape has n students from Huff. and n students from Rav. in his potions class. How many ways can he pick n students to do the lab?

LHS: Condition on how many are from Huff., $k = 0, 1, 2, \dots, n$.

$$\sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}$$

$\binom{n}{k}$: ways to pick k Huff.

$\binom{n}{n-k}$: ← ways to pick $n-k$ Rav.

$\{1, 2, \dots, n, n+1, \dots, 2n\}$ — how many of first n?

Sum of *squares* of binomial coefficients

Prove:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

2 Fast, 2 Furious

Prove: Sports team w/ n players, k starters, 2 co-captains (from starters)

$$\binom{n}{\ell}\cdot\binom{n-\ell}{k-\ell} = \binom{n}{k}\binom{k}{\ell}$$

$$\binom{n}{2}\cdot\binom{n-2}{k-2} = \binom{n}{k}\cdot\binom{k}{2}.$$

- $\binom{n}{2}$: pick 2 co-capt.s
- $\binom{n-2}{k-2}$: pick k-2 remaining starters.
- $\binom{n}{k}$: ways to pick k-comb. from set of size n
- $\binom{k}{2}$: ways to pick 2-comb. from set of size k.

Summations are Your Friends

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What is the summation variable?

Prove:

$$\sum_{r=0}^{n} \binom{n}{r} \binom{2n}{n-r} = \binom{3n}{n}.$$

Problem: How many ways to pick n of 3n distinct pieces of candy, where n contain choc. and 2n do not?

RHS: obvious.

LHS: r is # of choc. pieces we pick.

$\binom{n}{r}$ ways to pick r choc. ✓

$\binom{2n}{n-r}$ ways to pick n−r non-choc. ✓

$r = 0, 1, \ldots, n$.

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What is the summation variable?

Prove:

$$\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

→ # ways to pick r+1 distinct numbers from $\{1,2,3,\ldots,n+1\}$

What is k? One less than max picked.

$p_1, p_2, p_3, \ldots, p_{r+1}$. Assume sorted.

Condition on max number chosen.

If max = 1, then impossible.

If max = n+1, then $\binom{n}{r}$ ways.

If max = n: $\binom{n-1}{r}$

max = k+1: $\binom{k}{r}$

Recap: Learning Objective

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