Combinatorial Proofs

Ian Ludden

Learning Objective

By the end of this lesson, you will be able to:

Learning Objective

By the end of this lesson, you will be able to:

- Prove combinatorial identities by counting the same quantity in two ways.

Learning Objective

By the end of this lesson, you will be able to:

- Prove combinatorial identities by counting the same quantity in two ways.

NOTE: This is a special topic and will not be tested on any examlets.

What is a combinatorial proof?

What is a combinatorial proof?

Definition

A ***combinatorial proof*** is any argument that relies on counting.

What is a combinatorial proof?

Definition

A ***combinatorial proof*** is any argument that relies on counting.

We've seen this before...

For all $n, k \in \mathbb{N}$, prove

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

What is a combinatorial proof?

Definition

A ***combinatorial proof*** is any argument that relies on counting.

We've seen this before...

For all $n, k \in \mathbb{N}$, prove

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

General strategy to prove $A = B$:

1. Invent a counting problem you can solve in two ways.
2. Show that one answer to the counting problem is A.
3. Show that another answer is B.

BYO Word Problem

BYO Word Problem

Sum of binomial coefficients

Prove:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

BYO Word Problem

Sum of binomial coefficients

Prove:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Ascending/descending products

Prove:

$$1 \cdot n + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n \cdot 1 = \binom{n+2}{3}.$$

More Examples

More Examples

Sum of *squares* of binomial coefficients

Prove:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

More Examples

Sum of *squares* of binomial coefficients

Prove:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

2 Fast, 2 Furious

Prove:

$$\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}.$$

Summations are Your Friends

Summations are Your Friends

What is the summation variable?

Prove:

$$\sum_{r=0}^{n} \binom{n}{r} \binom{2n}{n-r} = \binom{3n}{n}.$$

Summations are Your Friends

What is the summation variable?

Prove:

$$\sum_{r=0}^{n} \binom{n}{r} \binom{2n}{n-r} = \binom{3n}{n}.$$

What is the summation variable?

Prove:

$$\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

Recap: Learning Objective

By the end of this lesson, you will be able to:

- Prove combinatorial identities by counting the same quantity in two ways.

NOTE: This is a special topic and will not be tested on any examlets.