Countability

Part b: To Infinity and Beyond

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Learning Objectives

By the end of this lesson, you will be able to:

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- Define countable and uncountable.

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- Recall standard examples of countable and uncountable sets.

Learning Objectives

By the end of this lesson, you will be able to:

- Define countable and uncountable.
- Recall standard examples of countable and uncountable sets.
- Identify whether a given set is countable or uncountable.

Countable Sets

Definition

An infinite set A is ***countably infinite*** if $|A| = |\mathbb{N}|$.

$a_0, a_1, a_2, a_3, \ldots$

$\mathbb{N}$, even $\mathbb{Z}$, $\mathbb{Z}$,
$\mathbb{Q}$, $\mathbb{N}^2$, password,
powers of 2, ...

Countable Sets

Definition

An infinite set A is ***countably infinite*** if $|A| = |\mathbb{N}|$.

Definition

A set is ***countable*** if it is finite or countably infinite.

$|A| = n \quad , n \in \mathbb{N}$

$a_0, a_1, \ldots, a_{n-1}$

$a_1, a_2, \ldots, a_n$

$a_0, a_1, \ldots, \ldots$

$a_n = f(n) \quad \forall n \in \mathbb{N}$

$|\mathbb{N}| = |\mathbb{Z}^+|$

$f(n) = n+1$

Countable Sets

Definition

An infinite set A is ***countably infinite*** if $|A| = |\mathbb{N}|$.

Definition

A set is ***countable*** if it is finite or countably infinite.

Fact

If B is countable and $A \subseteq B$, then A is countable.

Case 1: B is finite. Then $A \subseteq B$ is finite, so A is countable

Case 2: B is countably infinite, $\exists$ bij. $f: \mathbb{N} \to B$,

can prove $\exists$ bij. $g: \mathbb{N} \to A$ (Cantor-Schroeder-Bernstein)

Uncountable Sets

Uncountable Sets

Definition

A set S is ***uncountable*** if it is not countable.

Uncountable Sets

Definition

A set S is ***uncountable*** if it is not countable.

Do these even exist...?

$\mathbb{N}$, $\mathbb{Q}$

Cantor's Diagonalization Argument

Cantor's Diagonalization Argument

Theorem

$\mathbb{P}(\mathbb{N})$ *is uncountable.* set of all subsets of $\mathbb{N}$

Proof: by contradiction. Suppose $\mathbb{P}(\mathbb{N})$ is countably infinite.

Then $\exists$ bij. $g: \mathbb{N} \to \mathbb{P}(\mathbb{N})$.

$g(n)$ is a subset of $\mathbb{N}$.

We'll get a contradiction by showing g is not onto.

Build set $S \subseteq \mathbb{N}$ as follows:

$\forall k \in \mathbb{N}$, $k \in S$ iff $k \notin g(k)$.

Then $S \neq g(n)$ $\forall n \in \mathbb{N}$. So S has no pre-image. $\square$

$|\mathbb{N}| \leq |\mathbb{P}(\mathbb{N})|$: $f: \mathbb{N} \to \mathbb{P}(\mathbb{N})$, $f(n) = \{n\}$

Cantor's Diagonalization Argument

Theorem

$\mathbb{P}(\mathbb{N})$ *is uncountable.*

contains 0? 1? 2?

		b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	$\dots$	
$g(0)$	v_0	1	1	0	1	1	0	1	1	1	1	$\dots$	$\{0,1,3,4,6,7,8,9,\dots\}$
$g(1)$	v_1	1	1	0	0	1	0	1	1	0	0	$\dots$	$\{0,1,4,6,7,\dots\}$
$g(2)$	v_2	0	0	0	0	1	0	0	1	0	0	$\dots$	
	v_3	0	1	1	1	1	0	1	0	0	0	$\dots$	
	v_4	0	0	0	0	1	1	1	0	1	1	$\dots$	
	v_5	1	1	1	0	1	0	1	0	0	1	$\dots$	
	$\dots$	$\dots$											

sets in $\mathbb{P}(\mathbb{N})$

$v^* = 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ \dots$

$\forall i \in \mathbb{N},\ v^* \neq v_i$ (disagrees in position i)

Cantor's Diagonalization Argument

Theorem

$\mathbb{P}(\mathbb{N})$ *is uncountable.*

	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	...
v_0	1	1	0	1	1	0	1	1	1	1	...
v_1	1	1	0	0	1	0	1	1	0	0	...
v_2	0	0	0	0	1	0	0	1	0	0	...
v_3	0	1	1	0	1	0	1	0	0	0	...
v_4	0	0	0	0	1	1	1	0	1	1	...
v_5	1	1	1	0	1	0	1	0	0	1	...
...	...										

$v^* = 2 \quad 5 \quad 1 \quad 9 \quad 3 \quad 5$

Theorem

The interval $(0, 1)$ *of real numbers is uncountable.* $\mathbb{R}$, $\mathbb{R}^2$

More Uncountable Sets

More Uncountable Sets

Fact

If A is uncountable and $A \subseteq B$, then B is uncountable.

If B is countable and $A \subseteq B$, then A is countable.

More Uncountable Sets

Fact

If A is uncountable and $A \subseteq B$, then B is uncountable.

Theorem

The set of functions from $\mathbb{Z}$ to $\mathbb{Z}$ is uncountable.

subset: A = set of fns. from $\mathbb{Z}$ to $\{0,1\}$.

$h: \mathbb{N} \to \mathbb{Z}$

$f \in A$, then $g = f(h(n))$ is a fn. from $\mathbb{N}$ to $\{0,1\}$.

B = set of fns. from $\mathbb{N}$ to $\{0,1\}$. $|A| = |B|$.

$|B| = |\mathcal{P}(\mathbb{N})|$, so B is uncountable. $\rightarrow$ $|A|$ is uncountable

And $A \subseteq \{f : f: \mathbb{Z} \to \mathbb{Z}\}$, so $\{f : f: \mathbb{Z} \to \mathbb{Z}\}$ is uncountable. $\square$

Recap: Learning Objectives

$\mathbb{N}^2$, $\mathbb{Z}$, strings from some set of chars.

By the end of this lesson, you will be able to:

$(0,1)$, $[0,1]$, $\mathbb{R}$, $f: \mathbb{Z} \to \mathbb{Z}$, ...

- Define countable and uncountable.
- Recall standard examples of countable and uncountable sets.
- Identify whether a given set is countable or uncountable.