Countability

Part a: Extending Cardinality

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Learning Objectives

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- Construct a bijection between two sets to prove they have the same cardinality.
- Connect cardinality relationships to the existence of one-to-one functions.

Our Old Cardinality is Broken

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Our definition of cardinality from Chapter 5:

Given a set A, $|A|$ is the number of different objects in A.

$|\{2,3,4\}| = 3$ $\quad$ $|\mathbb{R}| = \infty$

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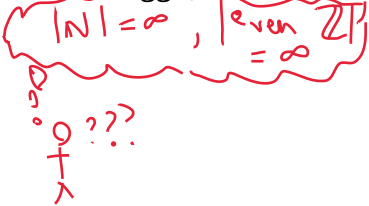
Which is bigger: $\mathbb{N}$, or the set of all even integers?



$\mathbb{N} \not\subseteq \text{even } \mathbb{Z}$

$\text{even } \mathbb{Z} \not\subseteq \mathbb{N}$

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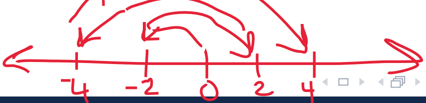
So $|\mathbb{N}| = |\{n \in \mathbb{Z} : n \text{ is even}\}|$, because:

$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ -n-1 & \text{if } n \text{ is odd} \end{cases}$$

n	$f(n)$
0	0
1	−2
2	2
3	−4
4	4
⋮	



Example 1: Prove $|\mathbb{N}| = |\{\text{powers of two}\}|$

Define $T = \{n \in \mathbb{Z} : n \geq 1 \text{ and } n \text{ is a power of two}\}$. Prove $|\mathbb{N}| = |T|$.

$g(n) = 2^n$. ✓

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Example 2: Prove $|\mathbb{Z}^+| = |\{\text{bit strings with no leading zeros}\}|$

Define $S = \{\text{bit strings with no leading zeros}\}$. Prove $|\mathbb{Z}^+| = |S|$.

$b(n)$ = binary representation of n

$b(1) = 1$, $b(2) = 10$, $b(14) = 1110$, ...

An Easier Way

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Definition

$|A| \leq |B| \leftrightarrow \exists$ one-to-one $f : A \rightarrow B$

$\{1,2,3\} \rightarrow \{a,b,c,d\}$

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Remember proving $A = B$ by proving $A \subseteq B$ and $B \subseteq A$?
Or $f(n)$ is $\Theta(g(n))$ by proving $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$?

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Theorem (Cantor-Schroeder-Bernstein)

Given sets A and B, if there exist one-to-one functions $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \to B$.

$(|A| \leq |B| \wedge |B| \leq |A| \to |A| = |B|)$

- Gives two-way bounding approach for proving $|A| = |B|$

Example 3: Prove $|\mathbb{N}^2| = |\mathbb{N}|$

Use the Cantor-Schroeder-Bernstein Theorem to prove $|\mathbb{N}^2| = |\mathbb{N}|$.

$\alpha : \mathbb{N}^2 \to \mathbb{N}$

$\alpha(n,m) = 2^n 3^m$

$\alpha(0,0) = 1, \alpha(4,1) = 48, \ldots$

$\beta : \mathbb{N} \to \mathbb{N}^2$

$\beta(n) = (n, 0)$

$|\mathbb{N}^2| \leq |\mathbb{N}|$

$|\mathbb{N}| \leq |\mathbb{N}^2|$

$|\mathbb{N}^2| = |\mathbb{N}|$. □

Example 3: Prove $|\mathbb{N}^2| = |\mathbb{N}|$

Use the Cantor-Schroeder-Bernstein Theorem to prove $|\mathbb{N}^2| = |\mathbb{N}|$.

a → 01
b → 02
c → 03
⋮
z → 26

$|\mathbb{N}| = |\{\text{passwords}\}|$

$|\mathbb{N}| \leq |\{\text{passwords}\}|$ and $|\{\text{passwords}\}| \leq |\mathbb{N}|$

Example 4: Prove $|\mathbb{N}| = |\{\text{passwords of lowercase letters}\}|$

Use the Cantor-Schroeder-Bernstein Theorem to prove $|\mathbb{N}| = |\{\text{passwords of lowercase letters}\}|$.

$c : \mathbb{N} \to \{\text{passwords}\}$

$c(n) = zzz\ldots z$ (n repetitions)

$d : \{\text{passwords}\} \to \mathbb{N}$

d(password) = encoding of password

d("datboi") = Ø40,120,02 1,509

Recap: Learning Objectives

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