Countability

Part a: Extending Cardinality

Ian Ludden

Learning Objectives

By the end of this lesson, you will be able to:

Learning Objectives

By the end of this lesson, you will be able to:

- Formally define what it means for two sets to have the same cardinality.

Learning Objectives

By the end of this lesson, you will be able to:

- Formally define what it means for two sets to have the same cardinality.
- Construct a bijection between two sets to prove they have the same cardinality.

Learning Objectives

By the end of this lesson, you will be able to:

- Formally define what it means for two sets to have the same cardinality.
- Construct a bijection between two sets to prove they have the same cardinality.
- Connect cardinality relationships to the existence of one-to-one functions.

Our Old Cardinality is Broken

Our Old Cardinality is Broken

Our definition of cardinality from Chapter 5:

Given a set A, $|A|$ is the number of different objects in A.

Our Old Cardinality is Broken

Our definition of cardinality from Chapter 5:

Given a set A, $|A|$ is the number of different objects in A.

Which is bigger: $\mathbb{N}$, or the set of all even integers?

Our Old Cardinality is Broken

Our definition of cardinality from Chapter 5:

Given a set A, $|A|$ is the number of different objects in A.

Which is bigger: $\mathbb{N}$, or the set of all even integers?

Definition

$|A| = |B| \leftrightarrow \exists$ bijection $f : A \to B$

Our Old Cardinality is Broken

Our definition of cardinality from Chapter 5:

Given a set A, $|A|$ is the number of different objects in A.

Which is bigger: $\mathbb{N}$, or the set of all even integers?

Definition

$|A| = |B| \leftrightarrow \exists$ bijection $f : A \rightarrow B$

So $|\mathbb{N}| = |\{n \in \mathbb{Z} : n \text{ is even}\}|$, because:

Proving Cardinality by Constructing Bijections

Example 1: Prove $|\mathbb{N}| = |\{\text{powers of two}\}|$

Define $T = \{n \in \mathbb{Z} : n \geq 1$ and n is a power of two$\}$. Prove $|\mathbb{N}| = |T|$.

Proving Cardinality by Constructing Bijections

Example 1: Prove $|\mathbb{N}| = |\{\text{powers of two}\}|$

Define $T = \{n \in \mathbb{Z} : n \geq 1 \text{ and } n \text{ is a power of two}\}$. Prove $|\mathbb{N}| = |T|$.

Example 2: Prove $|\mathbb{Z}^+| = |\{\text{bit strings with no leading zeros}\}|$

Define $S = \{\text{bit strings with no leading zeros}\}$. Prove $|\mathbb{Z}^+| = |S|$.

An Easier Way

An Easier Way

Definition

$|A| \leq |B| \leftrightarrow \exists$ one-to-one $f : A \rightarrow B$

An Easier Way

Definition

$|A| \leq |B| \leftrightarrow \exists$ one-to-one $f : A \rightarrow B$

Remember proving $A = B$ by proving $A \subseteq B$ and $B \subseteq A$?
Or $f(n)$ is $\Theta(g(n))$ by proving $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$?

An Easier Way

Definition

$|A| \leq |B| \leftrightarrow \exists$ one-to-one $f : A \rightarrow B$

Remember proving $A = B$ by proving $A \subseteq B$ and $B \subseteq A$?
Or $f(n)$ is $\Theta(g(n))$ by proving $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$?

Theorem (Cantor-Schroeder-Bernstein)

Given sets A and B, if there exist one-to-one functions $f : A \rightarrow B$ and $g : B \rightarrow A$, then there exists a bijection $h : A \rightarrow B$.

$(|A| \leq |B| \wedge |B| \leq |A| \rightarrow |A| = |B|)$

- Gives two-way bounding approach for proving $|A| = |B|$

Applying the Cantor-Schroeder-Bernstein Theorem

Example 3: Prove $|\mathbb{N}^2| = |\mathbb{N}|$

Use the Cantor-Schroeder-Bernstein Theorem to prove $|\mathbb{N}^2| = |\mathbb{N}|$.

Applying the Cantor-Schroeder-Bernstein Theorem

Example 3: Prove $|\mathbb{N}^2| = |\mathbb{N}|$

Use the Cantor-Schroeder-Bernstein Theorem to prove $|\mathbb{N}^2| = |\mathbb{N}|$.

Example 4: Prove $|\mathbb{N}| = |\{\text{passwords of lowercase letters}\}|$

Use the Cantor-Schroeder-Bernstein Theorem to prove $|\mathbb{N}| = |\{\text{passwords of lowercase letters}\}|$.

Recap: Learning Objectives

By the end of this lesson, you will be able to:

- Formally define what it means for two sets to have the same cardinality.
- Construct a bijection between two sets to prove they have the same cardinality.
- Connect cardinality relationships to the existence of one-to-one functions.