Proof by Contradiction

Part b: More Examples

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Learning Objective

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- Write a proof by contradiction.

Example 1: Anything But Two

Prove $\forall a, b \in \mathbb{Z}, a^2 - 4b \neq 2.$

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Prove $\forall a, b \in \mathbb{Z}$, $a^2 - 4b \neq 2$.

Proof.

The proof is by contradiction. Suppose there exist integers a and b such that $a^2 - 4b = 2$.

$a^2 = 4b + 2$

$a^2 = 2(2b+1)$

So a^2 is even, which means a is even. Let $a = 2k$ where $k \in \mathbb{Z}$.

$a^2 - 4b = (2k)^2 - 4b$

$= 4k^2 - 4b$

$2 = 4(k^2 - b)$

$\Downarrow$

$1 = 2\underbrace{(k^2 - b)}_{\text{integer}}$

So 1 is even. $\Rightarrow\Leftarrow$.

This is a contradiction. Therefore, $\forall a, b \in \mathbb{Z}$, $a^2 - 4b \neq 2$. □

Example 2: An Irrational Inequality

Prove $\sqrt{5} + \sqrt{13} > \sqrt{34}$.

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Prove $\sqrt{5} + \sqrt{13} > \sqrt{34}$.

Proof.

The proof is by contradiction. Suppose $\sqrt{5} + \sqrt{13} \le \sqrt{34}$.

Square both sides:

$$5 + 2\sqrt{5 \cdot 13} + 13 \le 34$$

$$\sqrt{65} \le \frac{34 - 18}{2} = 8$$

$$65 \le 64.$$

Direct.

$$65 = \left(\frac{(\sqrt{5} + \sqrt{13})^2 - 18}{2}\right)^2 > 64$$

$$\frac{(\sqrt{5} + \sqrt{13})^2 - 18}{2} > 8$$

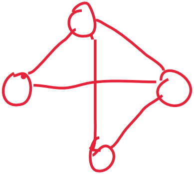
$$(\sqrt{5} + \sqrt{13})^2 > 34$$

$$\sqrt{5} + \sqrt{13} > \sqrt{34}$$

This is a contradiction. Therefore, $\sqrt{5} + \sqrt{13} > \sqrt{34}$. □

Example 3: Dense Graphs are Connected

Let $G = (V, E)$ be any graph with $|V| = n$. Prove that if every vertex in G has degree at least $n/2$, then G is connected.



Example 3: Dense Graphs are Connected

$\neg(p \to q) \to F \equiv (p \wedge \neg q) \to F$

Let $G = (V, E)$ be any graph with $|V| = n$. Prove that if every vertex in G has degree at least $n/2$, then G is connected. $p \to q$

Proof.

The proof is by contradiction. Suppose every vertex in G has degree at least $n/2$, but G is disconnected. Then G has $k \geq 2$ connected components, $G_1, G_2, \ldots, G_k$.

Let G_i be the conn. comp. with the fewest vertices.

Then G_i has $\leq \frac{n}{k} \leq \frac{n}{2}$ vertices. So degrees $\leq \frac{n}{2} - 1$.

But every vertex in G_i has degree at least $\frac{n}{2}$.

(but: same as "and")

This is a contradiction. Therefore, G is connected. □

Recap: Learning Objective

By the end of this lesson, you will be able to:

- Write a proof by contradiction.