Proof by Contradiction

Part a: The General Method

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Learning Objective

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- Outline a proof by contradiction.

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Indirect: prove p by showing $\neg p \rightarrow F$

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The more obviously false F is, the better. Examples:

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- $0 = 1$ $\quad$ $7 < 3$ $\quad$ 17 is a perf. square

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- $0 = 1$
- Purdue's marching band is better than the Marching Illini.

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2. "Suppose [negation of p]."
3. Take logical steps that end at a logical contradiction.
4. "This is a contradiction. Therefore, [p]."

A Worked Example

Prove $\sqrt{3}$ is irrational.

Proof.

The proof is by contradiction. Suppose $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ and $n \neq 0$. Suppose $\frac{m}{n}$ is in lowest terms ($\gcd(m,n)=1$).

$\sqrt{3} = \frac{m}{n}$ (mult. by n) $n\sqrt{3} = m$

↓ square both sides

$3 = \frac{m^2}{n^2} \longrightarrow 3n^2 = m^2$.

So $3 \mid (m^2)$, and $3 \mid m$. That is, $m = 3k$ for some $k \in \mathbb{Z}$.

Then $3n^2 = (3k)^2$

$3n^2 = 9k^2$

$n^2 = 3k^2$

So $3 \mid (n^2)$, and $3 \mid n$

So $\gcd(m,n) \geq 3 > 1$, ※. Hence

This is a contradiction. Therefore, $\sqrt{3}$ is irrational. $\sqrt{3}$ is irrational □

Recap: Learning Objective

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