Collections of Sets

Part c: Counting

Ian Ludden

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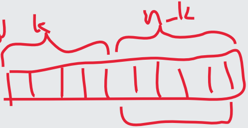
The number of k-combinations from a set of size n is

$$\frac{n!}{k!(n-k)!},$$

written $C(n, k)$ or $\binom{n}{k}$ and read as "n choose k."



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$$\frac{n!}{(n-k)!\,(n-(n-k))!}$$

$k!$

- Useful identity: $\binom{n}{k} = \binom{n}{n-k}$

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- Useful identity: $\binom{n}{k} = \binom{n}{n-k}$
- $P(n, k) = C(n, k) \cdot k!$

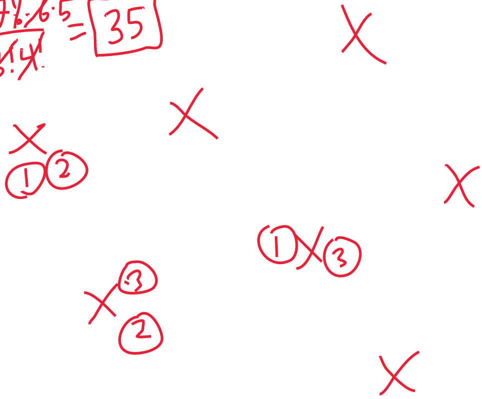
↑ order matters   ↑ # ways to order the chosen k elements

Applications of Combinations

Example 1: Facility Location

Given a set S of seven candidate locations for your new restaurant chain, how many ways are there to pick three starting locations?

$$\binom{7}{3} = \frac{7!}{3!\,4!} = \frac{7 \cdot 6 \cdot 5}{3!} = \boxed{35}$$



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Given a set S of seven candidate locations for your new restaurant chain, how many ways are there to pick three starting locations?

Example 2: Tennis Tournament

There are 16 players in a tennis tournament that must be split into two subtournaments of eight players each to play at separate stadiums. How many ways are there to pick which players are in which subtournament? Assume neither players nor fans care which stadium they play at.

stadium A — Stadium B

A	B
1–8	9–16
9–16	1–8

$$\frac{\binom{16}{8}}{2} = \frac{16!}{2 \cdot 8! \cdot 8!}$$

Combinations with Repetition

Example 3: Stock Portfolio

Suppose you're building a stock portfolio. There are three companies you are interested in, and you want to buy a total of 11 shares. How many different stock portfolios can you build? (Assume each company has at least 11 shares available to buy.)

11, 0, 0

3, 6, 2 $\neq$ 2, 6, 3

4, 2, 5

~~$\binom{11}{3}$~~

Combinations with Repetition

Example 3: Stock Portfolio

Suppose you're building a stock portfolio. There are three (n) companies you are interested in, and you want to buy a total of 11 (k) shares. How many different stock portfolios can you build? (Assume each company has at least 11 shares available to buy.)

Stars and bars: $* * * * \mid * * \mid * * * * *$

4, 2, 5

stars rep. shares

bars rep. boundaries b/w companies

xxx | xxxxxx | xx

xxxxxxxxxxx ||

$$\binom{11+2}{11} = \frac{(11+2)!}{11! \cdot 2!}$$

$$\binom{k+(n-1)}{k} = \binom{k+(n-1)}{n-1}$$

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

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- Proof: Look at whether you take a specific element

LHS: # k-combinations from a set A of n+1 elems

Consider a specific element $a \in A$.

If a is included, $\binom{n}{k-1}$ options.

If a is excluded, $\binom{n}{k}$ options.

Pascal's Identity

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- Can be used to build Pascal's triangle (Wikipedia link)

Pascal's Identity

$C(n,k) = \binom{n}{k}$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

- Proof: Look at whether you take a specific element
- Can be used to build Pascal's triangle (Wikipedia link)
- Recursively defines binomial coefficients (base: $k = 0, n$)

$\binom{n}{0} = 1$ $\emptyset$

$\binom{n}{n} = 1$ the whole set.

The Binomial Theorem

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Theorem

Let x and y be variables. For any natural number n,

"binomial"

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

$(x+y)^2 = x^2 + 2xy + y^2$

$\binom{2}{0} = 1$, $\binom{2}{1} = 2$,

$\binom{2}{2} = 1$.

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- Proof sketch: Pick which factors contribute a y.
  $(x+y)(x+y)(x+y)(\ldots)(x+y)$

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- Proof sketch: Pick which factors contribute a y.
  $(x+y)(x+y)(x+y)(\ldots)(x+y)$
- What is the coefficient of x^4y^2 in $(2x+y)^6$?

$a = 2x$

$(a+y)^6 = \ldots + \binom{6}{2} a^4 y^2 + \ldots$

$\binom{6}{2} = \frac{6 \cdot 5}{2!} = 15.$

$15 a^4 y^2 = 15 \cdot (2x)^4 \cdot y^2 = 240 x^4 y^2.$

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