Collections of Sets

Part c: Counting

Ian Ludden

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Given a set A with $|A| = n$ and an integer $0 \leq k \leq n$, how many subsets of A have size k?

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- $P(n,k) = C(n,k) \cdot k!$

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Example 2: Tennis Tournament

There are 16 players in a tennis tournament that must be split into two subtournaments of eight players each to play at separate stadiums. How many ways are there to pick which players are in which subtournament? Assume neither players nor fans care which stadium they play at.

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Example 3: Stock Portfolio

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Stars and bars: $* * * * \mid * * \mid * * * * *$

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- Can be used to build Pascal's triangle (Wikipedia link)
- Recursively defines binomial coefficients (base: $k = 0, n$)

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- What is the coefficient of x^4y^2 in $(2x+y)^6$?

Recap: Learning Objectives

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