Collections of Sets

Part a: Sets Containing Sets

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Learning Objectives

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- Interpret set-builder definitions of sets containing other sets.

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- Interpret set-builder definitions of sets containing other sets.
- Interpret definitions for functions whose input and/or output values are sets (i.e., when the domain and/or co-domain is a power set).

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- Manipulate sets containing other sets using standard set operations.
- Define the power set $\mathbb{P}(A)$ for a set A.
- Given a specific set A, list the elements of its power set.
- Interpret set-builder definitions of sets containing other sets.
- Interpret definitions for functions whose input and/or output values are sets (i.e., when the domain and/or co-domain is a power set).
- Write simple proofs involving collections of sets and/or functions whose input and/or output values are sets.

Sets of Sets

$\mathbb{Z} = \{0, 1, 2, -1, -2, \ldots\}$

$S = \{a, b, c\}$

$T = \{\{a, b\}, \{a, c\}, \{b, c\}\}$

Set<Integer>

"Set<Set>"

Set<Object>

Example 1

$A = \{☺, ☹, \{0, 2, 4\}, \{1, 3\}, \mathbb{Z}\}$ $|A| = 5$

$B = \{0, \mathbb{Z}^+, ☺, \{2, 4\}, \{1, 3\}\}$ $|B| = 5$

Example 1

$A = \{\cancel{☺}, ☹, \{0, 2, 4\}, \cancel{\{1, 3\}}, \mathbb{Z}\}$
$B = \{0, \mathbb{Z}^+, ☺, \{2, 4\}, \cancel{\{1, 3\}}\}$

- $A - B = \{a \in A : a \notin B\} = \{☹, \{0, 2, 4\}, \mathbb{Z}\}$

Sets of Sets

Example 1

$A = \{\smiley, \cancel{\frownie}, \cancel{\{0, 2, 4\}}, \{1, 3\}, \cancel{\mathbb{Z}}\}$
$B = \{\cancel{0}, \cancel{\mathbb{Z}^+}, \smiley, \cancel{\{2, 4\}}, \{1, 3\}\}$

- $A - B =$
- $A \cap B = \{\smiley, \{1, 3\}\}$

Sets of Sets

Example 1

$A = \{☺, ☹, \{0, 2, 4\}, \{1, 3\}, \mathbb{Z}\}$ $|A| = 5$

$B = \{0, \mathbb{Z}^+, ☺, \{2, 4\}, \{1, 3\}\}$ $|B| = 5$

- $A - B =$
- $A \cap B =$
- $|A \times B| =$ 25

We have to go deeper

"A ~~dream~~ set within a ~~dream~~ set within a ~~dream~~ set"

We have to go deeper

"A ~~dream~~ set within a ~~dream~~ set within a ~~dream~~ set"

Setception

(Inception Image Source)

Sets of sets of sets of...

Sets of sets of sets of...

Example 2

$A = \{\{1,4\}, \{5,7,8\}, \{3\}, \{\emptyset, \{2\}\}\}$ $|A| = 4$

$B = \{\{3,6\}, \emptyset, \{\emptyset, 2\}\}$ $|B| = 3$

Sets of sets of sets of...

Example 2

$A = \{\cancel{\{1,4\}}, \cancel{\{5,7,8\}}, \cancel{\{3\}}, \underline{\{\emptyset, \{2\}\}}\}$
$B = \{\{3,6\}, \emptyset, \underline{\cancel{\{\emptyset, 2\}}}\}$

- $A \cap B = \emptyset$

Sets of sets of sets of...

Example 2

$A = \{\{1,4\}, \{5,7,8\}, \{3\}, \{\emptyset, \{2\}\}\}$
$B = \{\{3,6\}, \emptyset, \{\emptyset, 2\}\}$

- $A \cap B = \emptyset$
- $B - A = B$

Sets of sets of sets of...

Example 2

$A = \{\{1,4\}, \{5,7,8\}, \{3\}, \{\emptyset, \{2\}\}\}$
$B = \{\{3,6\}, \emptyset, \{\emptyset, 2\}\}$

- $A \cap B =$
- $B - A =$
- $|A| = 4$

Power Sets

Definition

Given a set A, the ***power set*** of A, denoted $\mathbb{P}(A)$, is the set of all subsets of A. That is,

$$\mathbb{P}(A) = \{S : S \subseteq A\}.$$

Example 3

$A = \{1, 3, 4\}$
$B = \{1, 2, 3, 5\}$

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Example 3

$A = \{1, 3, 4\}$ $\quad 2 \cdot 2 \cdot 2 = 8$

$B = \{1, 2, 3, 5\}$

- $\mathbb{P}(A) = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\}$

$|\mathbb{P}(A)| = 8 = 2^3 = 2^{|A|}$

$A \in \mathbb{P}(A)$

$\mathbb{P}(A) = 2^A$

$\emptyset \in \mathbb{P}(A)$

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$A = \{1, 3, 4\}$
$B = \{1, 2, 3, 5\}$

- $\mathbb{P}(A) = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\}$
- $\mathbb{P}(A) \cap \mathbb{P}(B) = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\} = \mathbb{P}(A \cap B)$

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Example 3

$A = \{1, 3, 4\}$
$B = \{1, 2, 3, 5\}$

Is $\emptyset \subseteq \emptyset$? Yes!
$\forall e \in \emptyset$, $e \in \emptyset$.

- $\mathbb{P}(A) = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{3,4\}, \{1,3,4\}\}$
- $\mathbb{P}(A) \cap \mathbb{P}(B) =$
- $\mathbb{P}(\emptyset) =$ $\{\emptyset\}$

Power Sets as Function (Co-)Domains

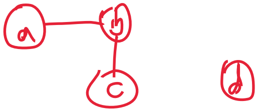
Example 4

Given a graph $G = (V, E)$, let $n : V \to \mathbb{P}(V)$ such that $n(v)$ is the set of vertices adjacent to v (not counting v, since we assume no self-loops).

$n(b) = \{a, c\}$

$n(d) = \emptyset$

$n(c) = n(a) = \{b\}$.



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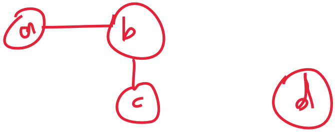
Example 5

Given a graph $G = (V, E)$, let $c : V \to \mathbb{P}(V)$ such that $c(v) = \{u \in V : \text{dist}(u, v) \leq 2\}$.

$c(a) = \{a, b, c\}$

$c(d) = \{d\}$

$c(b) = \{a, b, c\}$



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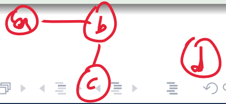
Example 6

Given a graph $G = (V, E)$, let $f : \mathbb{P}(V) \to \mathbb{N}$ such that for $S \subseteq V$, $f(S) = |\{uv \in E : u, v \in S\}|$.

$f(\{a,b\}) = |\{ab\}| = 1$

$f(\{a,b,c\}) = |\{ab, bc\}| = 2$

$f(\{b,d\}) = |\emptyset| = 0$



Example 7

Let $g : \mathbb{Z}^+ \to \mathbb{P}(\mathbb{Z}^+)$ such that $g(n) = \{q \in \mathbb{Z}^+ : n \mid q\}$.

g(n) is the set of (positive) multiples of n.

Example 7

Let $g : \mathbb{Z}^+ \to \mathbb{P}(\mathbb{Z}^+)$ such that $g(n) = \{q \in \mathbb{Z}^+ : n \mid q\}$.
Suppose $\underbrace{g(a)}_{\text{mult. of } a} = \underbrace{\underbrace{g(2)}_{\text{mult. of } 2} \cap \underbrace{g(7)}_{\text{mult. of } 7}}_{\text{mult. of } 14}$. What is a? Justify your answer.

$\boxed{a = 14}$.

Pf. ($\subseteq$) Let $n \in g(14)$ be arb. Then $14 \mid n$, so $2 \mid n$ and $7 \mid n$.

($\supseteq$) Let $m \in (g(2) \cap g(7))$ be arb.
Then $2 \mid m$ and $7 \mid m$, so $14 \mid m$.