Big-O

Part c: (Dis)Proving Big-O Relationships

Ian Ludden

Learning Objectives

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- Given functions f and g, (dis)prove that f is $O(g)$ and/or $\Theta(g)$.

Example 0: Warm-up

Prove $17n$ is $O(n^2)$.

$\underbrace{\exists c, k \in \mathbb{R}^+}_{\text{Fix some particular values}}$ $\underbrace{\forall n \geq k}_{\text{Prove universal claim}}$ $|f(n)| \leq c\,|g(n)|$.

$0 \leq f(n) \leq c g(n)$

Scratch:

$17n \leq c n^2 \quad \forall n \geq k$

$17 \leq cn$

$c = 1$

$k = 20$

$c = 17$
$k = 1$

$17n \leq 17n^2$
$= c \cdot n^2$.

Proof. Fix $c = 17$ and $k = 1$.
Let $n \geq k$ be arbitrary.
Then,

$17n \leq 17 n^2$ (since $n \geq k = 1$)
$= c \cdot n^2$.

So $17n$ is $O(n^2)$. $\square$

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Prove $17n$ is $O(n^2)$.

Example 1: The bigger, the better

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Prove $450n^2 + 25n + 2$ is $O(n^3)$.

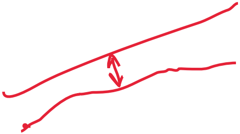
Scratch:

$450n^2 + 25n + 2 \le c \cdot n^3 \quad \forall n \ge k$

$450n^3 + 25n^3 + 2n^3 \le c \cdot n^3$

$c = 500$

$k = 1$



Proof: Fix $c = 500$, $k = 1$.

Let $n \ge k$ be arb.

Then,

$450n^2 + 25n + 2 \le 450n^3 + 25n^3 + 2n^3$ (since $n \ge k = 1$)

$= 477n^3$

$\le 500n^3$

$= c \cdot n^3$.

So $450n^2 + 25n + 2$ is $O(n^3)$.

$\square$

Example 2: I've heard it both ways

f is $O(g)$ and
g is $O(f)$ $\rightarrow$ f is $\Theta(g)$
and
g is $\Theta(f)$.

Example 2: I've heard it both ways

Prove $\overbrace{2n\log n + 3n}^{f}$ is $\Theta(\overbrace{n\log n}^{g})$.

Scratch. $2n\log n + 3n$ is $O(n\lg n)$

$2n\log n + 3n\underbrace{\log n}_{k \geq 2} = 5n\log n$

$c = 5$

Pf. Fix $c=5$, $k=2$.

Let $n \geq k$ be arb.

Then,

$$2n\log n + 3n \leq 2n\log n + 3n\log n \quad (\text{since } n \geq k = 2)$$
$$= 5n\log n$$
$$= c\cdot n\cdot\log n. \quad \square$$

Scratch. $n\log n$ is $O(2n\lg n + 3n)$

$c=1$, $k=1$

Pf. Fix $c=1$, $k=1$.

Let $n \geq k$ be arb.

Then,

$$n\log n \leq 2n\log n$$
$$\leq 2n\log n + 3n$$
$$= c\cdot(2n\log n + 3n). \quad \square$$

Example 3: Trading places

Prove $|\sin(n\pi/2)|$ is ***not*** $O(|\cos(n\pi/2)|)$.

$\exists c,k \in \mathbb{R}^+ \quad \forall n \geq k \quad |f(n)| \leq c|g(n)|$

$\underbrace{\forall c,k \in \mathbb{R}^+}\quad \underbrace{\exists n \geq k}\quad \underbrace{|f(n)| > c|g(n)|}$

Proof. Let $c,k \in \mathbb{R}^+$ be arb.

Construct $n = 2m+1$, where $m \in \mathbb{Z}$ and $m \geq k$.

Then,

n is odd   n is odd.

$$\left|\sin\left(\frac{n\pi}{2}\right)\right| = 1 > 0 = c \cdot \left|\cos\left(\frac{n\pi}{2}\right)\right|.$$

So $\left|\sin\left(\frac{n\pi}{2}\right)\right|$ is not $O\left(\left|\cos\left(\frac{n\pi}{2}\right)\right|\right)$.

$\square$

Scratch.

Need $n \geq k$, and n odd.

$n = 2m+1$, where $m \in \mathbb{Z}$ and $m \geq k$.

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