Big-O

Part a: Asymptotic Analysis of Functions

Ian Ludden

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$17n^2 + 5n + 4$ $3n^3 + 7$

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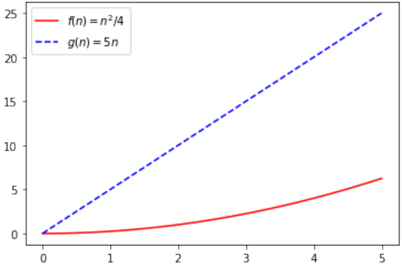

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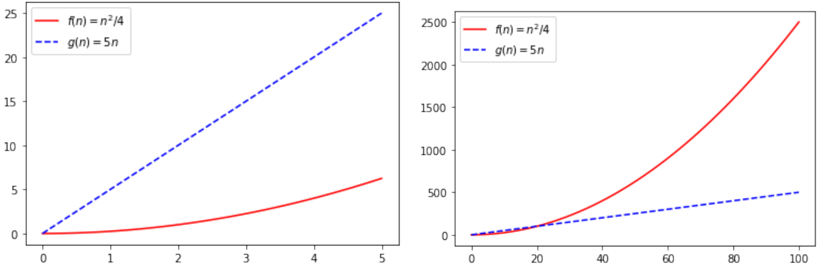

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Given functions $f, g : \mathbb{N} \to \mathbb{R}^+$, we say f and g are ***asymptotically similar*** ($f(n) \approx g(n)$) if (and only if)

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We say f is ***asymptotically smaller*** than g ($f(n) \ll g(n)$) if (and only if)

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Comparing Primitive Functions

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- $n^a \ll n^b$ whenever $b > a > 0$
- $n^k \ll 2^n$ for any $k \in \mathbb{Z}^+$

$\lim_{n \to \infty} \frac{n^k}{2^n} \quad \frac{k!}{2^n \cdots}$

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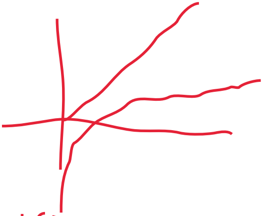
- $n^a \ll n^b$ whenever $b > a > 0$
- $n^k \ll 2^n$
- $a^n \ll b^n$ whenever $b > a > 0$

$$\lim_{n\to\infty} \frac{a^n}{b^n} = \lim_{n\to\infty} \left(\frac{a}{b}\right)^n$$

$<1 \to 0$

- $n^a \ll n^b$ whenever $b > a > 0$
- $n^k \ll 2^n$
- $a^n \ll b^n$ whenever $b > a > 0$
- $\log n \ll$ $n \ll n \log n \ll n^2$



$f(n) \ll g(n)$

$f(n) \cdot h(n) \ll g(n) \cdot h(n)$.

$$\lim_{n \to \infty} \frac{n \cdot 1}{n \log n} = 0$$

$$\lim_{n \to \infty} \frac{\log n}{n} = \lim_{n \to \infty} \frac{\frac{1}{n}}{1} = 0$$

$$\frac{n \log n}{n^2}$$

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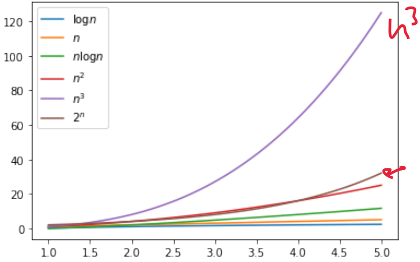

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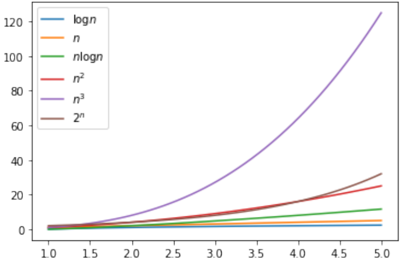

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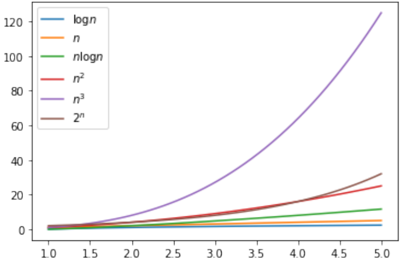

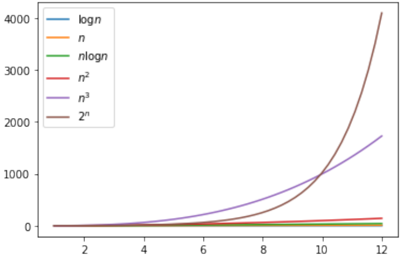

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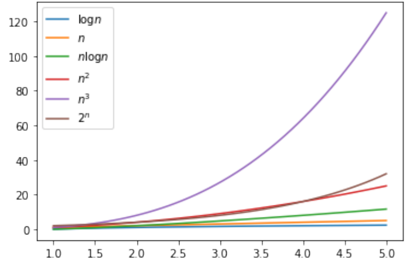
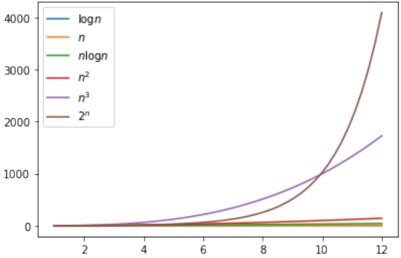
$2 \cdot 2 \cdot 2 \cdots 2$

$n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$





$$1 \ll \log n \ll n \ll n \log n \ll n^2 \ll n^3 \ll 2^n \ll 3^n \ll n!$$

$\lim_{n \to \infty} \frac{1}{\log n} = 0$

The Dominant Term Method

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Example 1

$f(n) = 13n + n! + 5$ versus $g(n) = 6^n + 4n^4 - 2$

$g \ll f$

$f \gg g$

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$h(n) = 15n \log n + 7n$ versus $r(n) = n^2/16 + 3 \log n$

$h(n) \ll r(n)$

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