Inductive Proofs with Grammar Trees

Ian Ludden

Learning Objective

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- Prove properties of context-free grammars by induction with parse trees.

Example 1: One More One

Let G be the grammar defined by start symbol S, terminals 0 and 1, and rules

$$S \to 0S1 \mid 1S0 \mid 1.$$

Prove by (strong) induction that all strings generated by G have more 1's than 0's. Use $\#0(T)$ and $\#1(T)$ as shorthand for the number of 0's and 1's in a tree T.

Pf: $\#1(T) > \#0(T)$.

The proof is by induction on h, the height of the parse tree.

Base Case: $h=1$. Then the parse tree is $T \equiv$ S | 1, which has $\#1(T) = 1 > 0 = \#0(T)$. ✓

Ind. step: Let $k > 1$ be arb. Suppose every tree generated by G with height $< k$ has more 1's than 0's.

Let T be an arb tree generated by G with height k.

Case 1:

Case 2:

Example 2: Odd Number of a's

Let G be the grammar defined by start symbol S, terminals $\{a, b\}$, and rules

$$S \to SabS \mid ab.$$

Prove by (strong) induction that all strings generated by G have an odd number of a's.

parse trees

Pf. The proof is by ind. on h, the height of the parse tree.

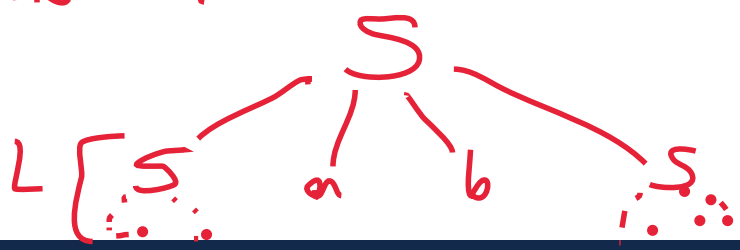
Base case: $h=1$. The only tree gen. by G with $h=1$ is

which has one a, and one is odd.

Ind. step: Let $k>1$ be arb. and suppose all trees gen. by G with height $<k$ have an odd number of a's.

Let T be an arb. tree gen. by G with height k.

The top of T looks like



Both L and R have $h \geq 1$ and $h \leq k-1$, so by I.H., L and R each have an odd # of a's.

So T has odd + 1 + odd = odd # a's.

Recap: Learning Objective

By the end of this lesson, you will be able to:

- Prove properties of context-free grammars by induction with parse trees.