More Recursion Trees

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Learning Objective

By the end of this lesson, you will be able to:

- Given a recursively defined function, find its closed form by drawing a recursion tree and adding up the work at all levels.

Example 1: Convenient Cancellation

Define $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ as

$$g(1) = 3$$
$$g(n) = 3g(n/3) + 2n \quad \forall n \geq 3.$$

Find a closed-form for $g(n)$ using a recursion tree. (Assume n is a power of 3.)

Example 1: Convenient Cancellation

Define $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ as

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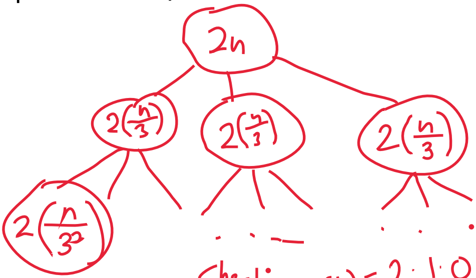
$g(n/3) + g(n/3) + g(n/3)$

Find a closed-form for $g(n)$ using a recursion tree. (Assume n is a power of 3.)

$g(3) = 3g(1) + 2(3) = 9 + 6 = 15$

Level	Input	Total Work
0	n	$2n$
1	$n/3$	$3 \cdot 2(\frac{n}{3}) = 2n$
2	$n/(3^2)$	$3^2 \cdot 2(\frac{n}{3^2}) = 2n$
k	$n/(3^k)$	$3^k \cdot 2(\frac{n}{3^k}) = 2n$
h	$n/(3^h) = 1$	$3^h \cdot 3 = 3n$



$h = \log_3 n$

$$g(n) = \sum_{k=0}^{h-1} (2n) + 3n$$
$$= 2n \cdot h + 3n$$
$$= \boxed{2n \log_3 n + 3n}$$

Check:

$g(1) = 2 \cdot 1 \cdot 0 + 3 \cdot 1 = 3$ ✓

$g(3) = 2(3)(1) + 3(3) = 15$ ✓

Example 2: Changing the Base Case

Let $S = \mathbb{Z}^+ - \{1\}$, and define $q : S \to \mathbb{Z}$ as

$$q(2) = 2$$
$$q(n) = 2q(n-1) + 5 \ \forall n \geq 3.$$

Find a closed-form for $q(n)$ using a recursion tree.

Let $S = \mathbb{Z}^+ - \{1\}$, and define $q : S \to \mathbb{Z}$ as

$$q(2) = 2$$

$$q(n) = 2q(n-1) + 5 \quad \forall n \geq 3.$$

Find a closed-form for $q(n)$ using a recursion tree.

$q(2) = 7 \cdot 2^{2-2} - 5 = 2$ ✓

$q(3) = 7 \cdot 2^{3-2} - 5 = 9$ ✓

$q(3) = 2q(2) + 5 = 2(2) + 5 = 9$

Level	Input	Total Work
0	n	5
1	$n-1$	$2 \cdot 5$
2	$n-2$	$2^2 \cdot 5$
k	$n-k$	$2^k \cdot 5$
h	$n-h=2$	$2^h \cdot 2 = 2^{n-1}$

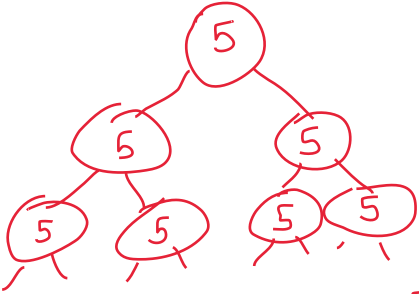
$h = n-2$

$$q(n) = \sum_{k=0}^{h-1} (2^k \cdot 5) + 2^{n-1}$$

$$= 5 \cdot \sum_{k=0}^{h-1} (2^k) + 2^{n-1}$$

$$= 5(2^h - 1) + 2^{n-1}$$

$$= 5 \cdot 2^{n-2} + 2^{n-1} - 5 = 7 \cdot 2^{n-2} -$$



Recap: Learning Objective

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