Introduction to Recursion Trees

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Learning Objective

- Given a recursively defined function, find its closed form by drawing a recursion tree and adding up the work at all levels.

Recursion trees are visualization tools.

Algorithm

```
function helloUniverse(n):
  if n = 1 do
        print("Hello, world!")
        return
  endif

  for i from 1 to n do
    print("Hello, world!")
  endfor

  helloUniverse(floor(n/2))
  helloUniverse(floor(n/2))
endfunction
```

base case

A silly recursive function

Recursive Definition

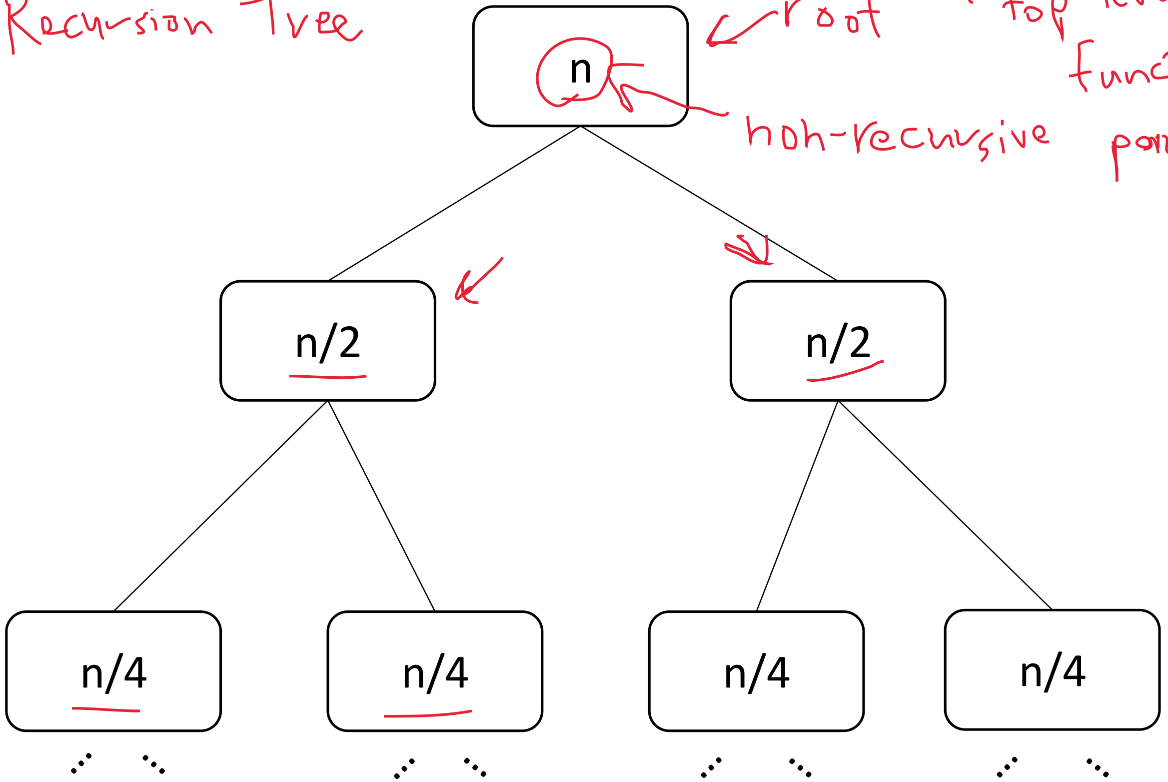
$$H(1) = 1$$

$$H(n) = \underbrace{n}_{\text{loop}} + 2 \cdot H\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \; \forall n \geq 2$$

non-rec. part

recursive calls

Recursion Tree



$$H(n) = n + 2\left(\frac{n}{2} + 2H\left(\frac{n}{4}\right)\right)$$

$$= n + 2 \cdot \left(\frac{n}{2}\right) + 4 \cdot H\left(\frac{n}{4}\right)$$

$$V(n) = \sum_{k=0}^{n-1}(n-k) + 1\cdot 1 = n\sum_{k=0}^{n-2}(1) - \left(\sum_{k=0}^{n-2} k\right) + 1 = n(n-1) - \left(\frac{(n-2)(n-1)}{2}\right) + 1 = \frac{2n^2 - 2n - (n^2-3n+2) + 2}{2}$$

$$= \frac{n^2 + n \cancel{+2 -2}}{2} = \boxed{\frac{n(n+1)}{2}}$$

Example 1: Are You Really a Tree?

unary

1-ary

vol = 1 in^3

Matryoshka nesting dolls

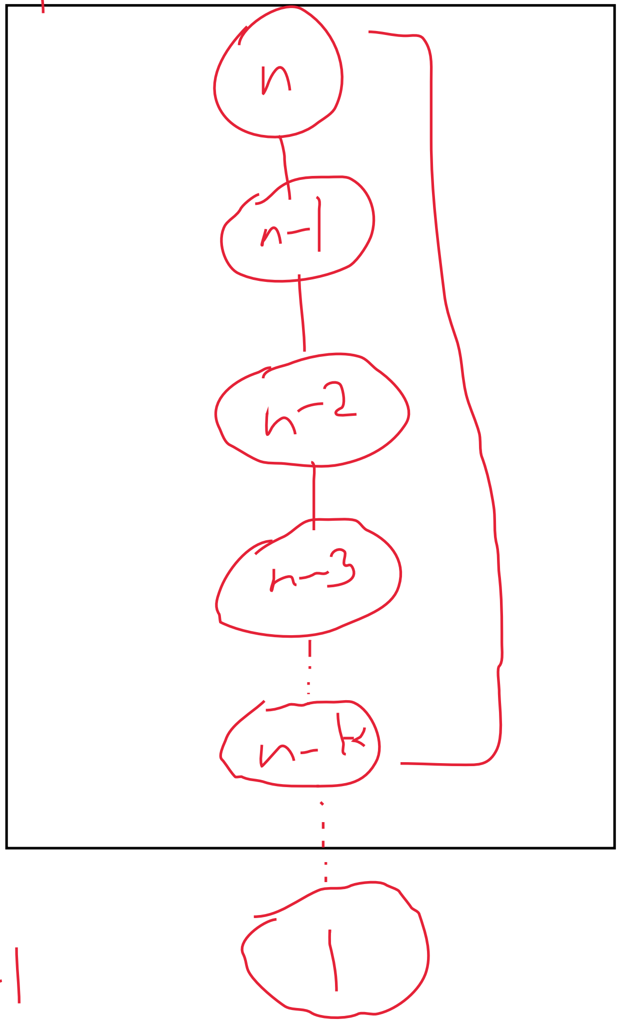
Source: https://images.app.goo.gl/aFx8iAvh5hhRpGeg9

Recursive definition of total volume (in^3):

V(1) = 1

V(n) = n + V(n-1)

Recursion Tree



$n - h = 1$

$h = n - 1$

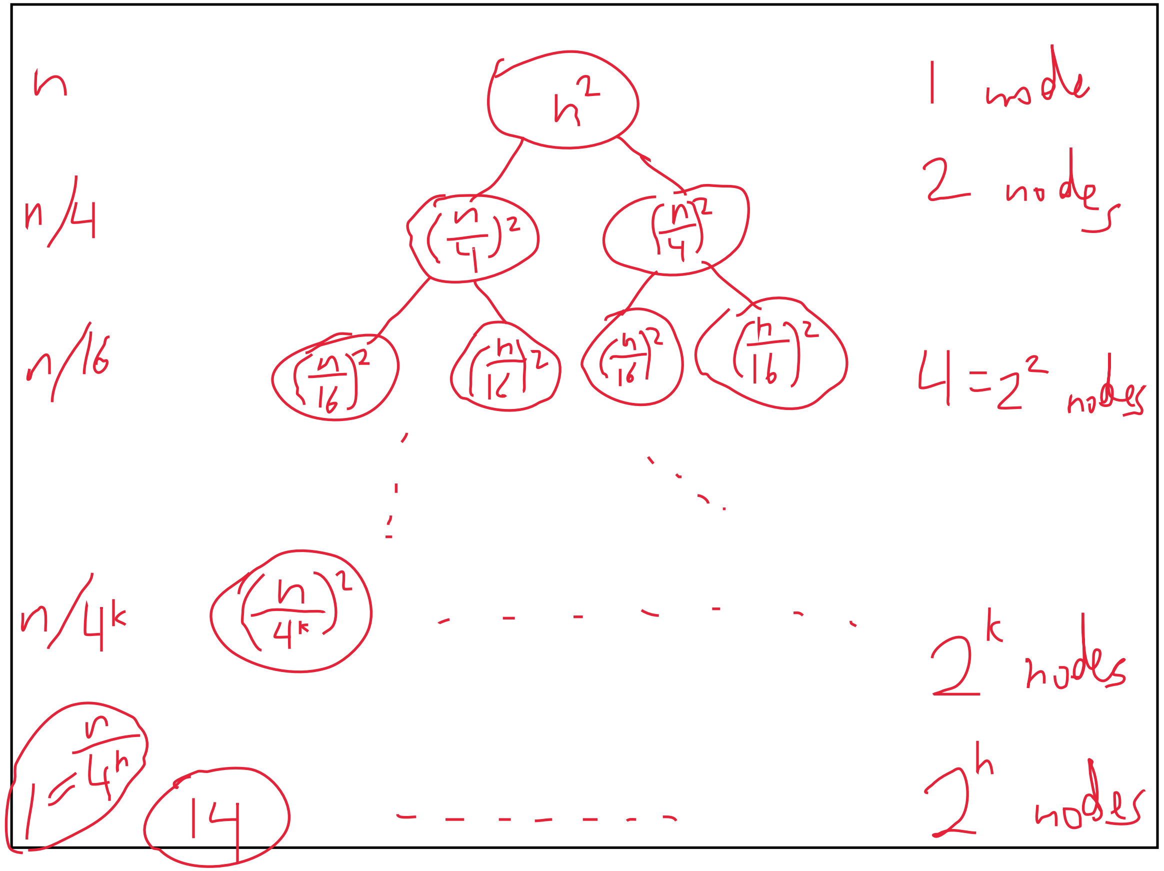
Level	Work "volume"
0	n
1	n-1
2	n-2
3	n-3
k	n-k
h	1

$2^{\log_4 n} = 2^{\log_2 \sqrt{n}} = \sqrt{n}$

Example 2: The ~~Plot~~ Tree Thickens

$f(1) = 14; \quad f(n) = 2 \cdot f\left(\frac{n}{4}\right) + n^2 \quad \forall n \geq 2$ (assume n is a power of 4)

$\frac{n}{4^h} = 1 \Rightarrow 4^h = n \Rightarrow h = \log_4 n$



Level	Work Per Node	Total Level Work
0	n^2	n^2
1	$\left(\frac{n}{4}\right)^2$	$2^1 \cdot \left(\frac{n}{4}\right)^2$
2	$\left(\frac{n}{16}\right)^2$	$2^2 \cdot \left(\frac{n}{4}\right)^2$
3	$\vdots$	$\vdots$
k	$\left(\frac{n}{4^k}\right)^2$	$2^k \cdot \left(\frac{n}{4^k}\right)^2$
h	14	$2^h \cdot 14$

$f(n) = \sum_{k=0}^{h-1} \left(2^k \cdot \left(\frac{n}{4^k}\right)^2\right) + 2^h \cdot 14$

$= n^2 \sum_{k=0}^{h-1} \left(\frac{1}{2^{3k}}\right) + 2^h \cdot 14$

$= n^2 \left(\frac{\left(\frac{1}{2^3}\right)^h - 1}{\frac{1}{2^3} - 1}\right) + 2^h \cdot 14$

$= n^2 \left(\frac{\left(\frac{1}{8}\right)^{h-1} - 8}{-7}\right) + 2^h \cdot 14$

$= n^2 \cdot \frac{1}{7}\left(8 - 2^{-3(h-1)}\right) + 2^h \cdot 14$

$= \frac{1}{7} n^2 \left(8 - 2^{-3(\log_4 n - 1)}\right) + 14 \cdot 2^{\log_4 n}$

$\boxed{14\sqrt{n} + \frac{8}{7} n^2 \left(1 - n^{-3/2}\right)}$

[Blank slide in case I needed more room]

Recap: Learning Objective

- Given a recursively defined function, find its closed form by drawing a recursion tree and adding up the work at all levels.