Tree Induction

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Learning Objective

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- Prove a claim about trees using induction.

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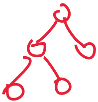
- I.H. for a particular value of the variable covers an entire *family* of trees $\forall h \in \mathbb{N}, P(h)$.

$\forall$ trees T of height h, $Q(T)$.



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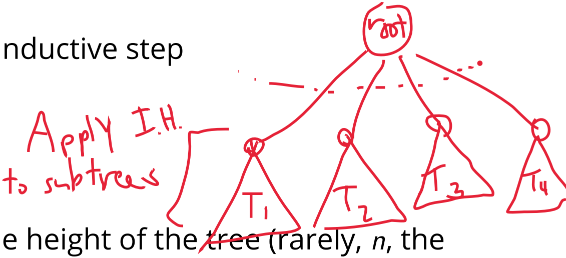


What's different?

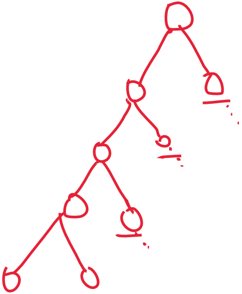
- Typically induct on h, the height of the tree (rarely, n, the number of nodes)
- I.H. for a particular value of the variable covers an entire *family* of trees
- Always divide tree up at the top (root plus subtrees of its children)

A Claim about Full Binary Trees

Recall: A binary tree with height h has $n \geq h+1$.

Claim

Let T be a full binary tree with height h and n nodes. Then $n \geq 2h+1$.



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Let T be a full binary tree with height h and n nodes. Then $n \geq 2h + 1$.

Proof: The proof is by induction on h, the height of the tree.

Base case: When $h = 0$, the tree is a single node, and $n = 1 \geq 1 = 2h + 1$. $n_T = 1 + n_X + n_Y \geq 1 + 2(k-1)+1 + 1$ $= 2k+1.$

Inductive step: Let $k > 0$ be an arbitrary natural number.

Suppose $n \geq 2h+1$ for all full binary trees of height $h < k$.

Suppose $\forall\ 0 \leq h < k$ that any full binary tree with height h and n nodes satisfies $n \geq 2h+1$.

(W.T.S.: Any full binary tree of height k has $\geq 2k+1$ nodes.)

Let T be an arbitrary full binary tree of height k with root v.

Since $k > 0$ and T is full, T has two children, x and y, with subtrees X and Y. WLOG, suppose X has height $k-1$. Then Y has height $0 \leq h_Y \leq k-1$.

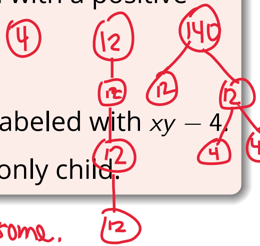
By I.H., X has $n_X \geq 2(k-1)+1$ nodes, and Y has $h_Y \geq 2(0)+1$ nodes.

Fearsome Trees

Definition

A ***fearsome*** tree is a binary tree with each node labeled with a positive integer, such that:

1. If v is a leaf node, then v is labeled with 4 or 12.
2. If v has two children with labels x and y, then v is labeled with $xy - 4$.
3. If v has one child, then v has the same label as its only child.



Obs. 1: Any subtree of a fearsome tree is fearsome.

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Proof: The proof is by induction on h, the height of the tree.
Base case: When $h = 0$, the tree is a single node with label 4 or 12. Both labels are congruent to 4 (mod 8).
Inductive step: Let $k > 0$ be an arbitrary natural number.
Suppose claim is true for all fearsome trees with height $0 \leq h < k$.

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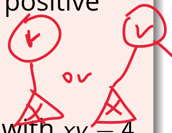

1. If v is a leaf node, then v is labeled with 4 or 12.
2. If v has two children with labels x and y, then v is labeled with $xy - 4$. [$\ell(x)\ell(y)$]
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Let T be an arb. fearsome tree with height k, and root r. $0 \le h$

There are two cases.

Case 1: r has two children, x and y w/ subtrees X and Y.

By Obs 1., X and Y are fearsome trees. So $\ell(x) \equiv 4 \pmod 8$ and $\ell(y) \equiv 4 \pmod 8$ by

Then, $\ell(r) = \ell(x)\ell(y) - 4$

$\equiv (4 \cdot 4) - 4 \pmod 8$

$\equiv 4 \pmod 8$.

Case 2: r has child x, $\ell(x)$, subtree X.

By I.H., $\ell(x) \equiv 4 \pmod 8$.

By defn, $\ell(r) = \ell(x)$, so $\ell(r) \equiv 4$

Recap: Learning Objective

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