Induction, Episode VI: Return of the I.H.

Part c: Proving Closed Forms by Induction

Ian Ludden

Learning Objective

By the end of this lesson, you will be able to:

- Use induction to prove facts about a recursively defined function, e.g., that it has some specific closed form.

Example 1: Our Old Friend Visits Again

Define $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$g(n) = \begin{cases} 2 & \text{if } n = 1 \\ n(n+1) + g(n-1) & \text{otherwise.} \end{cases}$$

Prove the closed-form expression for $g(n)$ is $\frac{n(n+1)(n+2)}{3}$.

Pf. The proof is by induction on n.

Base case: When $n=1$, $g(n)=2$ and $\frac{n(n+1)(n+2)}{3} = \frac{1(2)(3)}{3} = 2$

Ind. step: Let $k>1$ be arb. and suppose $g(n)= \frac{n(n+1)(n+2)}{3}$ $\forall\ 1 \le n < k$.

Then, $g(k) = k(k+1) + g(k-1)$. (by recursive formula)

$= k(k+1) + \frac{(k-1)(k)(k+1)}{3}$ (by I.H.)

$= \frac{3k^2+3k}{3} + \frac{k^3-k}{3}$

$= \frac{k^3+3k^2+2k}{3} = \frac{k(k+1)(k+2)}{3}$. ✓

Hence by induction, this closed form is correct $\forall n \in \mathbb{Z}^+$.

Example 2: Odd Fibonacci

Recall the Fibonacci sequence defined by

$$F(0) = F(1) = 1$$
$$F(n) = F(n-1) + F(n-2)\ \forall n \geq 2.$$

Prove $F(3n+1)$ is odd for all $n \in \mathbb{N}$.

Pf. The proof is by induction on n. Base case: When $n=0$, $F(3\cdot 0+1)=F(1)$ $=1$, odd[illegible]

Ind. step: Let $k \geq 0$ be arb. & suppose $F(3n+1)$ is odd $\forall\ 0 \leq n \leq k$.

then, $F(3(k+1)+1) = F(3k+4)$
$= F(3k+3) + F(3k+2)$
$= F(3k+2) + F(3k+1) + F(3k+2)$
$= 2\cdot F(3k+2) + F(3k+1)$

(even) (odd by I.H.)

$F(3k+2)+F(3k+1)$
$+F(3k+1)+F(3k)$

so $F(3(k+1)+1)$ is odd. Hence by induction, $F(3n+1)$ is odd $\forall$

Recap: Learning Objective

By the end of this lesson, you will be able to:

- Use induction to prove facts about a recursively defined function, e.g., that it has some specific closed form.