Induction, Episode VI: Return of the I.H.

Part b: Unrolling and Hypercubes

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Warning

If you haven't reviewed Section 1.5 of the textbook (Summations), now is a good time.

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$f(n) = \ldots + \ldots \cdot f(n-1)$

$= \ldots + \ldots(\ldots + \ldots f(n-2))$

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By Ludde Lorentz on Unsplash

Unrolling Practice

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Example 1: Implicit Summation

Define $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$g(n) = \begin{cases} 2 & \text{if } n = 1 \\ n(n+1) + g(n-1) & \text{otherwise.} \end{cases}$$

Find a closed-form expression for $g(n)$.

$$g(n) = \sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

$n-k=1$

$k = n-1$

$$g(n) = n(n+1) + g(n-1)$$
$$= n(n+1) + \big((n-1)(n) + g(n-2)\big)$$
$$= n(n+1) + (n-1)(n) + (n-2)(n-1) + g(n-3)$$
$$\vdots$$
$$= n(n+1) + (n-1)(n) + \dots + (n-k+1)(n-k+2) + g(n-k)$$
$$= \sum_{i=n-k+1}^{n} i(i+1) + g(n-k) = \sum_{i=2}^{n} i(i+1) + 2 = \sum_{i=1}^{n} i(i+1).$$

$$\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

Unrolling Practice

Example 2: Additive Terms

Define $b : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$b(1) = 3$$

$$b(n) = 3b(n-1) + 2n + 1 \;\forall n \geq 2.$$

$b(n-1) = 3b(n-2) + 2(n-1) + 1$

Find a closed-form expression for $b(n)$.

grouped first part of first sum with second sum.

$b(n) = 3[3b(n-2) + 2(n-1) + 1] + 2n + 1$

$= 3^2 b(n-2) + 3^1 2(n-1) + 3^0 2n + 3^1 1 + 3^0 1$

$= 3^k b(n-k) + \sum_{i=0}^{k-1} [3^i \cdot 2(n-i)] + \sum_{i=0}^{k-1} (3^i)$

When $n-k=1$, $k=n-1$.

$= 3^{n-1} \cdot b(1) + 2\sum_{i=0}^{n-2} [3^i (n-i)] + \sum_{i=0}^{n-2} (3^i)$

$= 3^{n-1} \cdot 3 + (2n+1)\sum_{i=0}^{n-2} (3^i) - 2 \cdot \sum_{i=0}^{n-2} (i \cdot 3^i)$

$= 3^n + (2n+1)\left(\frac{1-3^{n-1}}{1-3}\right) - 2\left[\frac{n \cdot 3^{n-1}}{2} - \frac{5 \cdot 3^{n-1}}{4} + \frac{3}{4}\right]$

⋮ (algebra)

$= \boxed{2 \cdot 3^n - n - 2}$

difficult... you will not be asked to find closed forms of similar sums

Hypercubes

Definition

The k-dimensional hypercube, Q_k, is a graph defined recursively for $n \in \mathbb{N}$ by

- Q_0 is a single vertex with no edges.
- For any $k \geq 1$, Q_k is two copies of Q_{k-1} with edges joining corresponding vertices.

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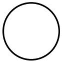
Q_0:



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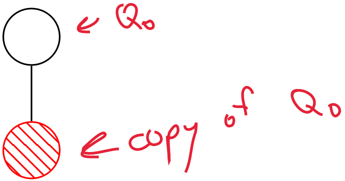

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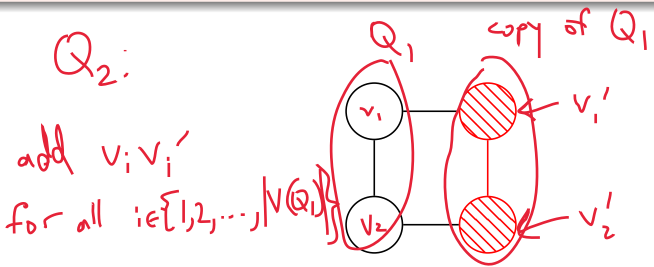

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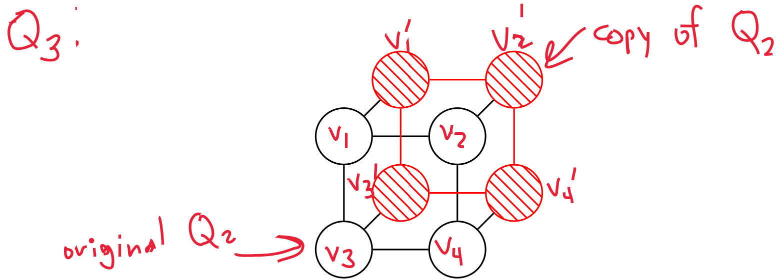

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$|V(Q_0)| = 1$

$|V(Q_1)| = 2$

$Q_2 \to 4$

$Q_3 \to 8$

$$V(k) = \begin{cases} 1 & \text{if } k = 0 \\ 2 \cdot V(k-1) & \text{if } k > 0 \end{cases}$$

How many vertices does Q_k have?

$$V(k) = 2 \cdot V(k-1)$$

$$= 2 \cdot (2 \cdot V(k-2))$$

$$= 2 \cdot (2 \cdot (2 V(k-3)))$$

$$\vdots$$

$$= 2^j \cdot V(k-j).$$

$k - j = 0$

$j = k$.

$$= 2^k \cdot V(0) = \boxed{2^k}.$$

Recap: Learning Objectives

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