Induction, Episode V: The Recursion Fairy Strikes Back

Part c: Induction Proofs of More Involved Claims

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Learning Objectives

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$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Example 1: Induction with Divides

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Claim

For all $a, b \in \mathbb{Z}^+$ *and* $n \in \mathbb{N}$, $a \mid b \to a^n \mid b^n$.

$2^{2n} \cdot 3^{3n} \mid 2^{4n} \cdot 3^{3n}$

Example 1: Induction with Divides

Claim

For all $a, b \in \mathbb{Z}^+$ *and* $n \in \mathbb{N}$, $a \mid b \to a^n \mid b^n$. $P(n)$.

Proof.

The proof is by induction on n. $P(n) := \forall a, b \in \mathbb{Z}^+, a \mid b \to a^n \mid b^n$.
Base case: We show $P(0)$. $a^0 = 1$, $b^0 = 1$, so $a^0 \mid b^0$ ✓.

Inductive step: Let $k > 0$ be arbitrary.
Suppose $P(n)$ is true for $0 \leq n < k$. $n = 0, 1, \ldots, k-1$

Example 1: Induction with Divides

Claim

For all $a, b \in \mathbb{Z}^+$ *and* $n \in \mathbb{N}$, $a \mid b \to a^n \mid b^n$.

$\underbrace{\forall n \in \mathbb{N}}_{\text{induction}}, \forall a, b \in \mathbb{Z}^+, a \mid b \to a^n \mid b^n.$

Proof.

The proof is by induction on n. $P(n) := \forall a, b \in \mathbb{Z}^+, \underbrace{a \mid b \to a^n \mid b^n}.$
Base case: We show $P(0)$.

$b^n = a^n \cdot m$

Inductive step: Let $k > 0$ be arbitrary.
Suppose $P(n)$ is true for $0 \leq n < k$.

Let $a, b \in \mathbb{Z}^+$ be arb., and suppose $a \mid b$.

Then $b^k = b^{k-1} \cdot b = b^{k-1} \cdot (a \cdot q) = (a^{k-1} \cdot m) \cdot (aq) = a^k (mq)$.

Hence $P(k)$ is true.

By induction, $P(n)$ is true for all $n \geq n_0$. $= 0$ □

Indexing our I.H.

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Claim

$\forall n \in \mathbb{N},\ n \geq n_0,\ P(n).$

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Claim

$\forall n \in \mathbb{N}, n \geq n_0, P(n)$.

Proof.

Base cases: We show $P(n_0), P(n_0+1), \ldots, P(n_1)$.

$n_0 \leq n_1$

Inductive step: Let $k > n_1$ be arbitrary.
Suppose $P(n)$ is true for $n_0 \leq n < k$.

$\leq k-1$

Indexing our I.H.

Claim

$\forall n \in \mathbb{N}$, $n \geq n_0$, $P(n)$.

Proof.

Base cases: We show $P(n_0), P(n_0+1), \ldots, P(n_1)$.

Inductive step: Let $k > n_1$ be arbitrary.
Suppose $P(n)$ is true for $n_0 \leq n \leq k$.

[Details] Hence $P(\overset{k+1}{\cancel{k}})$ is true.
↳ allowed to use $P(n_0), P(n_0+1), \ldots, P(k-1), P(k)$

By induction, $P(n)$ is true for all $n \geq n_0$. □

Example 2: Trigonometric Inequality

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Claim

For any $x \in \mathbb{R}$ *and* $n \in \mathbb{N}$, $|\sin(nx)| \leq n|\sin x|$.

$\cancel{1 \leq \frac{1}{2}|0|}$

$n = \frac{1}{2}, \quad x = \pi$

Example 2: Trigonometric Inequality

Claim

For any $x \in \mathbb{R}$ *and* $n \in \mathbb{N}$, $\underbrace{|\sin nx| \leq n|\sin x|}_{①}$. $\forall n \in \mathbb{N}, \forall x \in \mathbb{R},$ ①

Proof.

The proof is by induction on n. $P(n) := \forall x \in \mathbb{R}, |\sin nx| \leq n|\sin x|$.

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For any $x \in \mathbb{R}$ *and* $n \in \mathbb{N}$, $|\sin nx| \leq n|\sin x|$.

Proof.

The proof is by induction on n. $P(n) := \forall x \in \mathbb{R}, |\sin nx| \leq n|\sin x|$.

Base case: When $n = 0$, $|\sin 0x| = 0 \leq 0|\sin x| \quad \forall x \in \mathbb{R}. \checkmark$

Example 2: Trigonometric Inequality

$|\sin kx| = |\sin((k-1)x + x)|$

Claim

For any $x \in \mathbb{R}$ *and* $n \in \mathbb{N}$, $|\sin nx| \leq n|\sin x|$.

Proof.

The proof is by induction on n. $P(n) := \forall x \in \mathbb{R}, |\sin nx| \leq n|\sin x|$.

Base case: When $n = 0$,

Inductive step: Let $k \geq 0$ be arbitrary, and suppose $P(n)$ is true for $0 \leq n \leq k$. We show $P(k+1)$ is true:

$|a+b| \leq |a| + |b|$.

$$
\begin{aligned}
|\sin(k+1)x| &= |\sin(kx + x)| \\
&= |\sin kx \cos x + \sin x \cos kx| \\
&\leq |\sin kx \cos x| + |\sin x \cos kx| \\
&= |\sin kx| \cdot |\cos x| + |\sin x| \cdot |\cos kx| \\
&\leq \underbrace{|\sin kx|} + |\sin x| \\
&\leq \underbrace{k|\sin x|} + |\sin x| \\
&= (k+1)|\sin x|.
\end{aligned}
$$

Hence $P(k+1)$ is true.

Hence by induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Recap: Learning Objectives

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