Induction, Episode V: The Recursion Fairy Strikes Back

Part c: Induction Proofs of More Involved Claims

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Learning Objectives

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Base case: We show $P(0)$.

Inductive step: Let $k > 0$ be arbitrary.
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Hence $P(k)$ is true.

By induction, $P(n)$ is true for all $n \geq n_0$. □

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Base case: When $n = 0$,
Inductive step: Let $k \geq 0$ be arbitrary, and suppose $P(n)$ is true for $0 \leq n \leq k$. We show $P(k+1)$ is true:

Hence by induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Recap: Learning Objectives

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