Induction, Episode V: The Recursion Fairy Strikes Back

Part b: Picking Base Cases

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Learning Objectives

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- Explain why proofs with too few base cases break.

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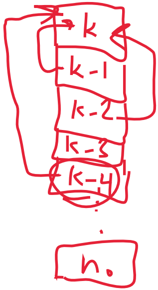
$n=8$ through $n=12$

$n=8$ " $n=10$

$k>10 \Rightarrow k \geq 11$

$8 \leq k-3 < k$

$k \geq n_0+4$



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- Example: Filling tour buses with groups of size 3 or 5
- **Q**: Can you have too few base cases?
- **A**: Yes! Watch out for this.

$8 = 3 + 5$ ✓

$9 = 3 \cdot 3$ ✓

Let $k \geq 9$ be arb. & suppose we can fill $n = 8$ through $n = k - 1$.

Put a group of 3 on.

By I.H., we can fill the remaining $k - 3$.

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We then have $k = (k-1) + 1$.

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irrational + rational = irrational

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Since 1 is rational and we know the sum of a rational and an irrational is irrational, k is irrational.

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$k = 0$

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Hence by induction, every natural number is irrational. □

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We then have

$$\begin{aligned} k &= (k-2) + 2 \\ &= 2m + 2 \quad \text{for some } m \in \mathbb{Z} \quad \text{(since } k-2 \text{ is even by the I.H.)} \\ &= 2(m+1), \end{aligned}$$

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Recap: Learning Objectives

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