Induction, Episode IV: A New Proof Technique

Part b: A Full, Slow Example

Ian Ludden

Learning Objectives

By the end of this lesson, you will be able to:

Learning Objectives

By the end of this lesson, you will be able to:

- Given a claim, identify/state the key parts of an inductive proof.

Learning Objectives

By the end of this lesson, you will be able to:

- Given a claim, identify/state the key parts of an inductive proof.
- Use induction to prove a formula is correct for all integers starting at some n_0.

Our First Inductive Proof Together

Claim

For every natural number n, $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

$$0^2 = 0 = \frac{0(0+1)(2\cdot 0+1)}{6}$$

$$0^2 + 1^2 = 1 = \frac{1(2)(3)}{6}$$

$$0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30 = \frac{2\cancel{4}(5)(\cancel{9})3}{\cancel{6}\,\cancel{3}\,1}$$

Our First Inductive Proof Together

Claim

For every natural number n, $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

$\forall n \in \mathbb{N}, P(n)$.

Proof.

We prove the claim by induction on n.

Let $P(n)$ be the statement $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Base case: We need to show $P(0)$ is true.

When $n=0$, LHS is $\sum_{i=0}^{0} i^2 = 0^2 = 0$, and

RHS is $\frac{0(1)(1)}{6} = 0$, so $P(0)$ is true.

Our First Inductive Proof Together

Claim

For every natural number n, $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof.

We prove the claim by induction on n.

Let $P(n)$ be the statement $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Base case: We need to show $P(0)$ is true.

Inductive step: Let $k > 0$ be arbitrary. Suppose that $P(n)$ is true for $n = 0, 1, \ldots, k-1$. (This is called the ***inductive hypothesis***, or I.H. for short.) We need to show that $P(k)$ is true.

$\forall k > n_0, \underbrace{\bigwedge_{n=n_0}^{k-1} P(n)}_{} \to P(k)$

We have $\sum_{i=0}^{k} i^2 = k^2 + \sum_{i=0}^{k-1} i^2$

$P(k-1)$ tells us $\sum_{i=0}^{k-1} i^2 = \underbrace{\frac{(k-1)(k-1+1)(2(k-1)+1)}{6}}_{①}$

$= k^2 + ①$

$= k^2 + \frac{2k^3 - 3k^2 + k}{6}$

$= \frac{2k^3 + 3k^2 + k}{6} = \frac{k(k+1)(2k+1)}{6}$

Hence $P(k)$ is true.

By induction, $P(n)$ is true $\forall n \in \mathbb{N}$. $\square$

Q.E.D.

$\square$

Full Proof

Proof.

We prove the claim by induction on n. $P(n) := \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
Base case: When $n = 0$, $\sum_{i=0}^{0} i^2 = 0^2 = \frac{0(0+1)(2\cdot 0+1)}{6}$, so $P(0)$ is true.
Inductive step: Let $k > 0$ be arbitrary. Suppose that $P(n)$ is true for $n = 0$ through $n = k - 1$. We have

$$\begin{aligned}
\sum_{i=0}^{k} i^2 &= k^2 + \sum_{i=0}^{k-1} i^2 \\
&= k^2 + \frac{(k-1)(k-1+1)(2(k-1)+1)}{6} && \text{(by the I.H.)} \\
&= \frac{2k^3 + 3k^2 + k}{6} && \text{(by algebra)} \\
&= \frac{k(k+1)(2k+1)}{6} && \text{(by factoring).}
\end{aligned}$$

applied to $n = k-1$

Hence $P(k)$ is true.
By induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Reminder: Parts of Inductive Proof

$$\left[\underbrace{S(n_0)}_{\text{Base case}} \wedge \overbrace{\left[\forall \underset{\text{pick arb.}}{k} > n_0 \left[\overbrace{\underbrace{\bigwedge_{n=n_0}^{k-1} S(n)}_{\text{assume I.H.}} \rightarrow \underbrace{S(k)}_{\text{show } S(k)}}^{\text{inductive step}} \right]\right]}^{\text{universal claim}}\right] \rightarrow \forall n \geq n_0\ S(n).$$

Learning Objectives

By the end of this lesson, you will be able to:

- Given a claim, identify/state the key parts of an inductive proof.
- Use induction to prove a formula is correct for all integers starting at some n_0.