Induction, Episode IV: A New Proof Technique

Part a: Mathematical Foundation and Helpful Analogies

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Learning Objectives

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- State the Well-Ordering Principle and the Principle of Mathematical Induction.

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- Explain induction to your smart ten-year-old cousin.

The Well-Ordering Principle

Fact (The Well-Ordering Principle)

*Every nonempty subset of $\mathbb{Z}^+$ contains a smallest element. (We say that $\mathbb{Z}^+$ is **well-ordered**.) This holds for $\mathbb{N}$ too.*

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$q \quad 0 < q/2 < q$

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- Not true for $\mathbb{Q}^+$, $\mathbb{R}^+$
- Interesting, but is it useful??

The Principle of Mathematical Induction

Deduction

Theorem

Let $S(n)$ denote a statement (logical expression) about n, where n can be replaced with any natural number. If

(a) *$S(0)$ is true (the* ***base case****), and*

(b) *For all $k \in \mathbb{Z}^+$, $\bigwedge_{n=0}^{k-1} S(0) \rightarrow S(k)$ (****the inductive step****),*

then $S(n)$ is true for all $n \in \mathbb{N}$.

$S(0) \wedge S(1) \wedge \ldots \wedge S(k-1) \rightarrow S(k)$

$\sum_{n=0}^{k-1}$

Theorem

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then $S(n)$ is true for all $n \in \mathbb{N}$.

Proof (sketch). Let $S(n)$ be a statement defined $\forall n \in \mathbb{N}$.
Suppose (a) and (b) hold.
Let $F = \{f \in \mathbb{N} : \neg S(f)\}$, we want to show $F = \emptyset$.

Suppose to the contrary that $F \neq \emptyset$.
Then by the well-ordering principle, F has a smallest element, m.
If $m = 0$, then $S(0)$ is false, but we assumed it's true. $\Rightarrow\Leftarrow$.
If $m > 0$, then $S(k)$ is true for all $0 \le k \le m-1$.
By (b), $S(m)$ is true. $\Rightarrow\Leftarrow$

p

$\neg p \to F$

$\neg(\neg p) \vee F$

p

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Proof (sketch).

More flexible version:

$$\left[S(n_0) \wedge \left[\forall k > n_0 \left[\bigwedge_{n=0}^{k-1} S(0) \to S(k) \right] \right] \right] \to \forall n \geq n_0 \; S(n).$$

For $n_0 \in \mathbb{Z}$: (a) (b) $\in \mathbb{Z}$ n n_0

Analogy 1: Dominos

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Analogy 2: Infinite Skyscraper

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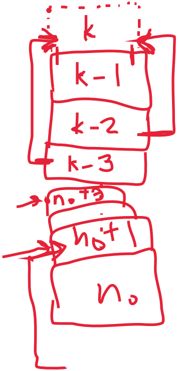

$S(k-3) \to S(k)$

← "ground floor" or base case

Recap: Learning Objectives

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