Induction, Episode IV: A New Proof Technique

Part a: Mathematical Foundation and Helpful Analogies

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Learning Objectives

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The Well-Ordering Principle

Fact (The Well-Ordering Principle)

Every nonempty subset of $\mathbb{Z}^+$ contains a smallest element. (We say that $\mathbb{Z}^+$ is ***well-ordered****.) This holds for $\mathbb{N}$ too.*

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- Not true for $\mathbb{Q}^+$, $\mathbb{R}^+$
- Interesting, but is it useful??

The Principle of Mathematical Induction

Theorem

Let $S(n)$ denote a statement (logical expression) about n, where n can be replaced with any natural number. If

(a) *$S(0)$ is true (the **base case**), and*

(b) *For all $k \in \mathbb{Z}^+$, $\bigwedge_{n=0}^{k-1} S(n) \rightarrow S(k)$ (**the inductive step**),*

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More flexible version:

$$\left[S(n_0) \land \left[\forall k > n_0 \left[\bigwedge_{n=n_0}^{k-1} S(n) \to S(k)\right]\right]\right] \to \forall n \geq n_0\ S(n).$$

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Analogy 2: Infinite Skyscraper

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Recap: Learning Objectives

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