Graph Coloring

Ian Ludden

Learning Objectives

By the end of this lesson, you will be able to:

Learning Objectives

By the end of this lesson, you will be able to:

- State the definitions/notation of graph coloring and chromatic number.

Learning Objectives

By the end of this lesson, you will be able to:

- State the definitions/notation of graph coloring and chromatic number.
- Apply upper and lower bounds to prove the chromatic number of a graph.

Definitions and Notation

Definition

- A k-***coloring*** of a graph $G = (V, E)$ is a function $h : V \to \{1, 2, \ldots, k\}$ such that $\forall uv \in E$, $h(u) \neq h(v)$.

S, $|S| = k$ proper k-coloring

Definitions and Notation

Definition

- A k-***coloring*** of a graph $G = (V, E)$ is a function $h : V \to \{1, 2, \ldots, k\}$ such that $\forall uv \in E$, $h(u) \neq h(v)$.
- If a k-coloring of G exists, then G is k-***colorable***.

Definitions and Notation

Definition

- A k-***coloring*** of a graph $G = (V, E)$ is a function $h : V \to \{1, 2, \ldots, k\}$ such that $\forall uv \in E$, $h(u) \neq h(v)$.
- If a k-coloring of G exists, then G is k-***colorable***.
- The ***chromatic number*** of G, $\chi(G)$, is the minimum integer k such that G is k-colorable.

\chi

Definitions and Notation

Definition

- A k-***coloring*** of a graph $G = (V, E)$ is a function $h : V \to \{1, 2, \ldots, k\}$ such that $\forall uv \in E$, $h(u) \neq h(v)$.
- If a k-coloring of G exists, then G is k-***colorable***.
- The ***chromatic number*** of G, $\chi(G)$, is the minimum integer k such that G is k-colorable.

Definitions and Notation

Definition

- A k-***coloring*** of a graph $G = (V, E)$ is a function $h : V \to \{1, 2, \ldots, k\}$ such that $\forall uv \in E$, $h(u) \neq h(v)$.
- If a k-coloring of G exists, then G is k-***colorable***.
- The ***chromatic number*** of G, $\chi(G)$, is the minimum integer k such that G is k-colorable.

Fact

If G is k-colorable, then $\chi(G) \leq k$. ← upper bound on $\chi(G)$.

Greedy Coloring

Greedy Coloring

- Order the vertices $v_1, v_2, \ldots, v_n$.

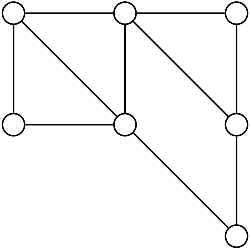
Greedy Coloring

- Order the vertices $v_1, v_2, \ldots, v_n$.
- For i from 1 to n:

Greedy Coloring

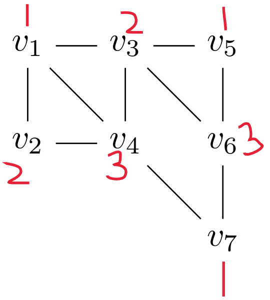
- Order the vertices $v_1, v_2, \ldots, v_n$.
- For i from 1 to n:
- Assign v_i the smallest positive integer (color) that hasn't been used yet by one of its neighbors.

- Order the vertices $v_1, v_2, \ldots, v_n$.
- For i from 1 to n:
- Assign v_i the smallest positive integer (color) that hasn't been used yet by one of its neighbors.



Greedy Coloring

- Order the vertices $v_1, v_2, \ldots, v_n$.
- For i from 1 to n:
- Assign v_i the smallest positive integer (color) that hasn't been used yet by one of its neighbors.



v_1: 1

v_2:

$\deg(v) :=$ # edges incident to v

In iteration i, we color v_i at most $\deg(v_i)+1$.

Max color used $\leq \max_{v \in V}\{\deg(v)\}+1$.

Greedy Coloring

- Order the vertices $v_1, v_2, \ldots, v_n$.
- For i from 1 to n:
- Assign v_i the smallest positive integer (color) that hasn't been used yet by one of its neighbors.

Lemma

If D is the maximum degree of a vertex in G, then the greedy coloring uses at most $D + 1$ colors.

$\chi(G) \leq D+1.$

Upper Bounds on $\chi(G)$

- A specific k-coloring of G proves $\chi(G) \leq k$.

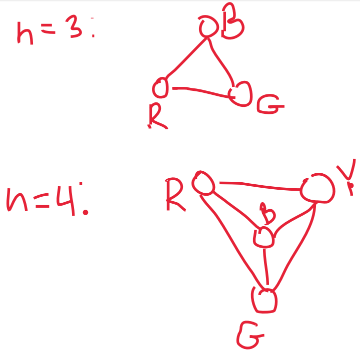
Upper Bounds on $\chi(G)$

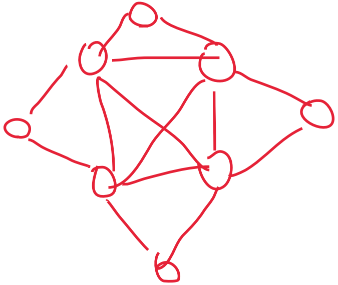
- A specific k-coloring of G proves $\chi(G) \leq k$.
- If the maximum degree in G is D, then G is $(D+1)$-colorable.

$\chi(G) \leq D+1$

Lemma

If G contains K_n as a subgraph, then $\chi(G) \geq n$.

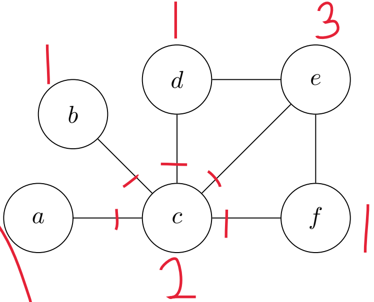




Example: Find and prove chromatic number

Greedy: $k=3$

$\chi(G) \leq 3$.



Max deg.: $D=5$

$\chi(G) \leq D+1 = 6$

cde and cfe form copies of K_3.

$\chi(G) \geq 3$

$3 \leq \chi(G) \leq 3$,

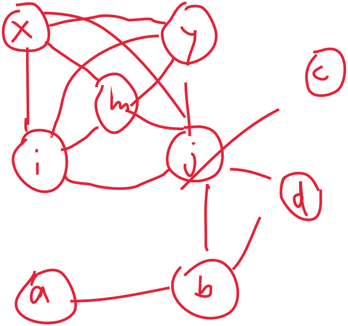
so $\chi(G) = 3$.

Applications

1. Register allocation when compiling C/C++/Java

```
int x = 0
int y = 1
for i = 1 to n:
    for j = i to n:
        int m = i + j
        int a = 2
        int b = 4
    int c = 1
    int d = 2
```

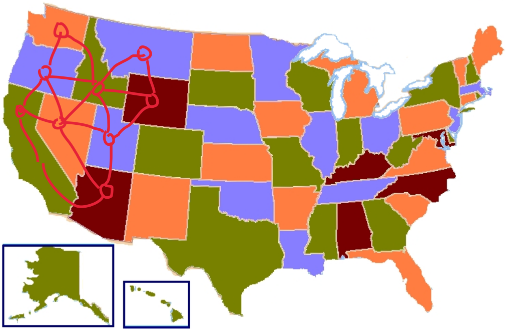
clique



Applications

1. Register allocation when compiling C/C++/Java
2. Coloring a political map (search "Four Color Theorem" online)

conjecture



Source: CC BY-SA 3.0, Wikimedia

1976

Appel Haken

Recap: Learning Objectives

By the end of this lesson, you will be able to:

- State the definitions/notation of graph coloring and chromatic number.
- Apply upper and lower bounds to prove the chromatic number of a graph.