Graph Coloring

Ian Ludden

Learning Objectives

By the end of this lesson, you will be able to:

Learning Objectives

By the end of this lesson, you will be able to:

- State the definitions/notation of graph coloring and chromatic number.

Learning Objectives

By the end of this lesson, you will be able to:

- State the definitions/notation of graph coloring and chromatic number.
- Apply upper and lower bounds to prove the chromatic number of a graph.

Definitions and Notation

Definition

- A k-***coloring*** of a graph $G = (V, E)$ is a function $h : V \to \{1, 2, \ldots, k\}$ such that $\forall uv \in E$, $h(u) \neq h(v)$.

Definitions and Notation

Definition

- A k-***coloring*** of a graph $G = (V, E)$ is a function $h : V \to \{1, 2, \ldots, k\}$ such that $\forall uv \in E$, $h(u) \neq h(v)$.
- If a k-coloring of G exists, then G is k-***colorable***.

Definitions and Notation

Definition

- A k-***coloring*** of a graph $G = (V, E)$ is a function $h : V \to \{1, 2, \ldots, k\}$ such that $\forall uv \in E$, $h(u) \neq h(v)$.
- If a k-coloring of G exists, then G is k-***colorable***.
- The ***chromatic number*** of G, $\chi(G)$, is the minimum integer k such that G is k-colorable.

Definitions and Notation

Definition

- A k-***coloring*** of a graph $G = (V, E)$ is a function $h : V \to \{1, 2, \ldots, k\}$ such that $\forall uv \in E$, $h(u) \neq h(v)$.
- If a k-coloring of G exists, then G is k-***colorable***.
- The ***chromatic number*** of G, $\chi(G)$, is the minimum integer k such that G is k-colorable.

Definitions and Notation

Definition

- A k-***coloring*** of a graph $G = (V, E)$ is a function $h : V \to \{1, 2, \ldots, k\}$ such that $\forall uv \in E$, $h(u) \neq h(v)$.
- If a k-coloring of G exists, then G is k-***colorable***.
- The ***chromatic number*** of G, $\chi(G)$, is the minimum integer k such that G is k-colorable.

Fact

If G is k-colorable, then $\chi(G) \leq k$.

Greedy Coloring

Greedy Coloring

- Order the vertices $v_1, v_2, \ldots, v_n$.

Greedy Coloring

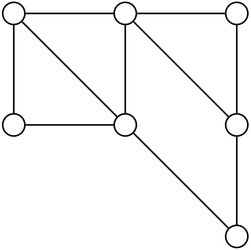
- Order the vertices $v_1, v_2, \ldots, v_n$.
- For i from 1 to n:

Greedy Coloring

- Order the vertices $v_1, v_2, \ldots, v_n$.
- For i from 1 to n:
- Assign v_i the smallest positive integer (color) that hasn't been used yet by one of its neighbors.

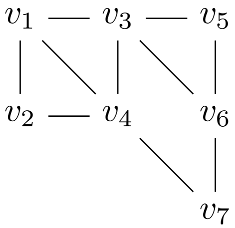
Greedy Coloring

- Order the vertices $v_1, v_2, \ldots, v_n$.
- For i from 1 to n:
- Assign v_i the smallest positive integer (color) that hasn't been used yet by one of its neighbors.



Greedy Coloring

- Order the vertices $v_1, v_2, \ldots, v_n$.
- For i from 1 to n:
- Assign v_i the smallest positive integer (color) that hasn't been used yet by one of its neighbors.



Greedy Coloring

- Order the vertices $v_1, v_2, \ldots, v_n$.
- For i from 1 to n:
- Assign v_i the smallest positive integer (color) that hasn't been used yet by one of its neighbors.

Lemma

If D is the maximum degree of a vertex in G, then the greedy coloring uses at most $D + 1$ colors.

Upper Bounds on $\chi(G)$

- A specific k-coloring of G proves $\chi(G) \leq k$.

Upper Bounds on $\chi(G)$

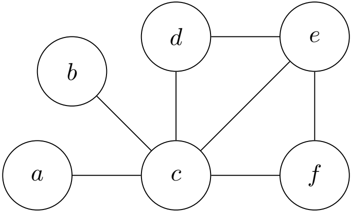
- A specific k-coloring of G proves $\chi(G) \leq k$.
- If the maximum degree in G is D, then G is $(D+1)$-colorable.

Lower Bound on $\chi(G)$

Lemma

If G contains K_n as a subgraph, then $\chi(G) \geq n$.

Example: Find and prove chromatic number

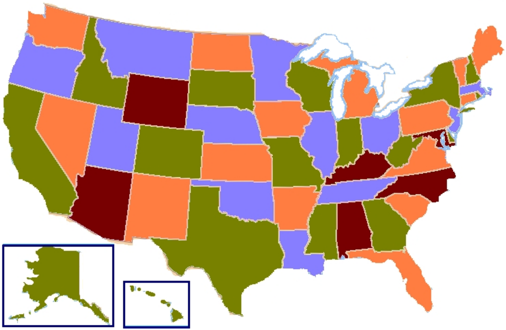


Applications

1. Register allocation when compiling C/C++/Java

Applications

1. Register allocation when compiling C/C++/Java
2. Coloring a political map (search "Four Color Theorem" online)



Source: CC BY-SA 3.0, Wikimedia

Recap: Learning Objectives

By the end of this lesson, you will be able to:

- State the definitions/notation of graph coloring and chromatic number.
- Apply upper and lower bounds to prove the chromatic number of a graph.