Proving Set Equality

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Learning Objective

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- Prove a set equality by proving inclusion in both directions.

$\shortparallel$

"subset"

Set Equality as Two-way Bounding

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Definition

Given sets A and B in a universe U, we say A and B are equal ($A = B$) if (and only if) $A \subseteq B$ and $B \subseteq A$.

$\forall a \in A, a \in B$ $\quad$ $\forall b \in B, b \in A.$

Set Equality as Two-way Bounding

Definition

Given sets A and B in a universe U, we say A and B are equal ($A = B$) if (and only if) $A \subseteq B$ and $B \subseteq A$.

- Can view as two-way bounding: A is no larger than B, and B is no larger than A

$a \leq b$ $b \leq a$

Example 1

Let $A = \{2m + 5 : m \in \mathbb{Z}\}$ and $B = \{2n - 3 : n \in \mathbb{Z}\}$. Prove $A = B$.

$-1, 1, 3, 5, 7, 9,$ $\quad$ $-5, -3, -1, 1, 3, 5$

Proof: ① $A \subseteq B$:

Let $a \in A$ be arb. That is, $a = 2m+5$ for some $m \in \mathbb{Z}$.

Then $a = 2m+5$

$= 2m + 8 - 3$

$= 2(m+4) - 3$

$= 2n - 3$, where $n = m+4 \in \mathbb{Z}$.

So $a \in B$, and we conclude $A \subseteq B$.

② $B \subseteq A$: Let $b \in B$ be arb. So $b = 2n-3$ for some $n \in \mathbb{Z}$.

$b = 2n-3 = 2n - 3 + 8 - 8 = (2n-8) + 5 = 2m+5$, where $m = n-4 \in \mathbb{Z}$.

Example 2

Let $S = \{x \in \mathbb{Z} : x \text{ is an odd multiple of } 3\}$ and $T = [3]_6$.
Prove $S = T$.

Proof.

① $S \subseteq T$:

Let $s \in S$ be arb.
Then $3 | s \Rightarrow s = 3 \cdot l$ and $s = 2k+1$ for some $k, l \in \mathbb{Z}$.

W.T.S. $6 | (s-3)$

~~Case 1: l is even.~~

Case 2: l is odd.

If l is even, then $s = 3 \cdot 2 \cdot m$
$= 6m, = 2 \cdot (3m)$
so $s \equiv 0 \mod 3$

Then l must be odd.

$s = 3(2n+1) = 6n+3$. $s-3 = 6 \cdot n$, so $6 | (s-3)$.

② $T \subseteq S$:

Let $t \in T$ be arb.
Then $6 | (t-3)$, so $t-3 = 6n, n \in \mathbb{Z}$.
So $t = 6n+3$.
Since $t = 2(3n+1)+1$, t is odd.
Also, $t = 3(2n+1)$, so $3 | t$.
Hence t is an odd multiple of 3. ($t \in S$).

Since $S \subseteq T$ and $T \subseteq S$, we conclude $S = T$. $\square$

Recap: Learning Objective

By the end of this lesson, you will be able to:

- Prove a set equality by proving inclusion in both directions.