Graph Isomorphisms

Ian Ludden

Learning Objectives

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- Prove that two graphs are (not) isomorphic.

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$G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$

$f : V_1 \to V_2$

function: one-to-one onto

$|V_1| = |V_2|$

$\forall a, b \in V_1, \quad ab \in E_1 \iff f(a)f(b) \in E_2$

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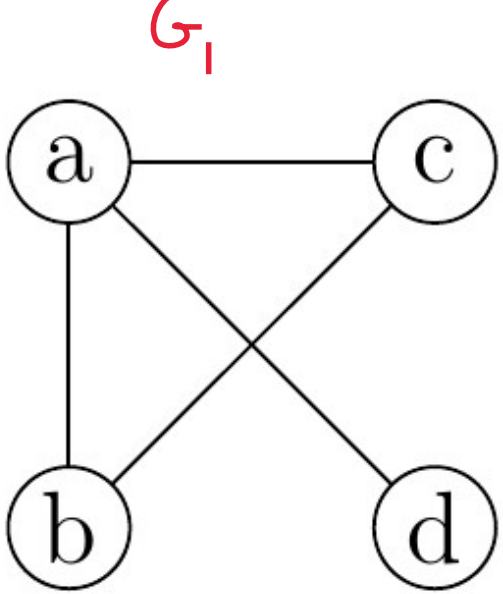
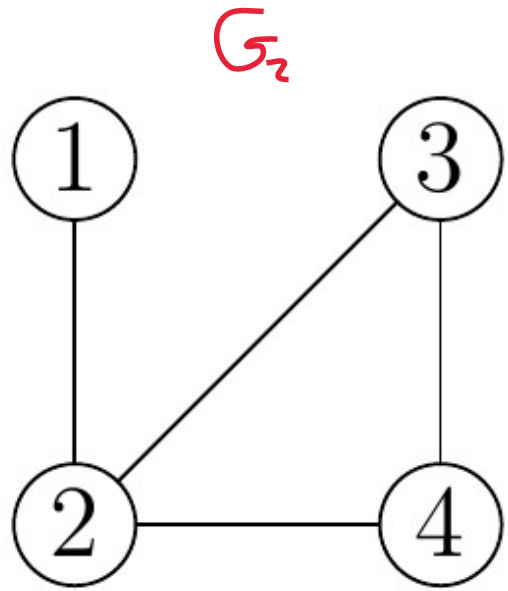
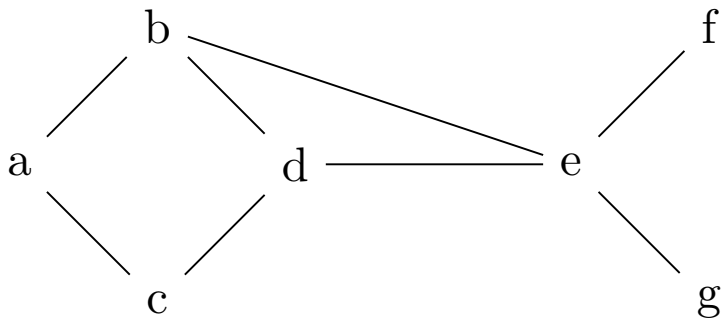




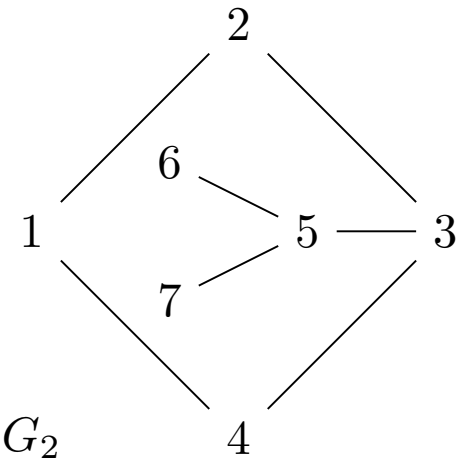
v	f(v)
a	2
b	3
c	4
d	1

Counting isomorphisms

$G_1 \sim G_2$ and $G_2 \sim G_3 \rightarrow G_1 \sim G_3$ R, S, T

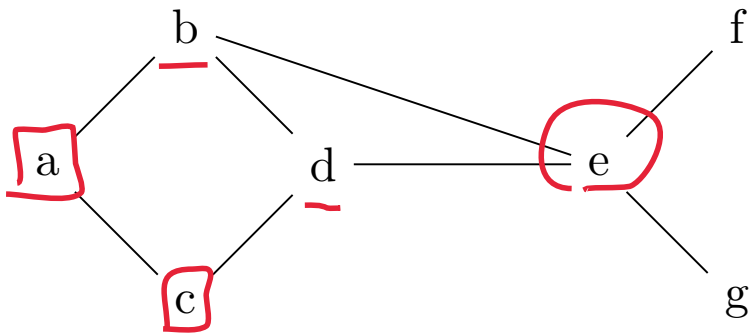


G_1



G_2

Counting isomorphisms



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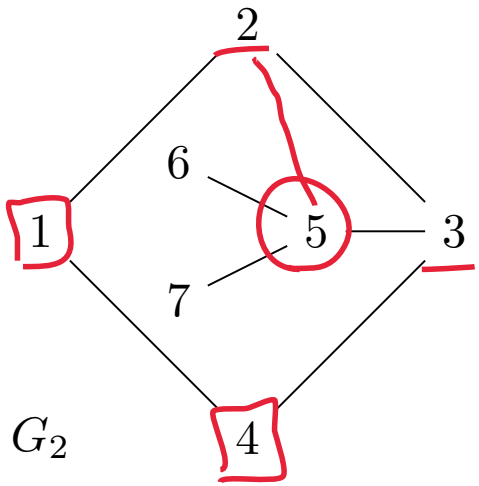


G_2

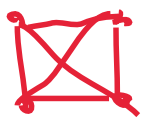
v	$f(v)$
a	determined by b/d (1 or 4)
b	2 or 3
c	
d	2 or 3
e	5
f	6 or 7
g	6 or 7

$2 \cdot 2 = 4$

f/g b/d

G_1 G_2

4!



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$\forall f : V_1 \to V_2$, f is not an isomorphism.

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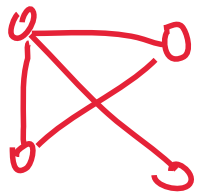

1: 1
2: 2
3: 1

1,2,2,3

1: 3
2: 0
3: 1

1,1,1,3

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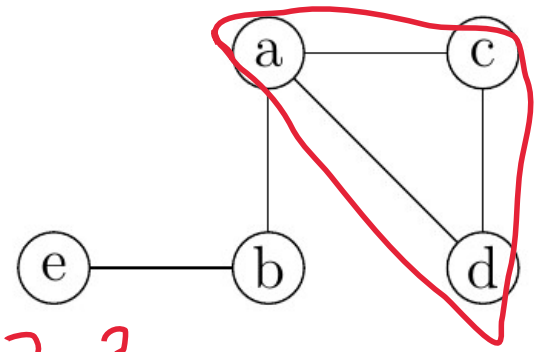
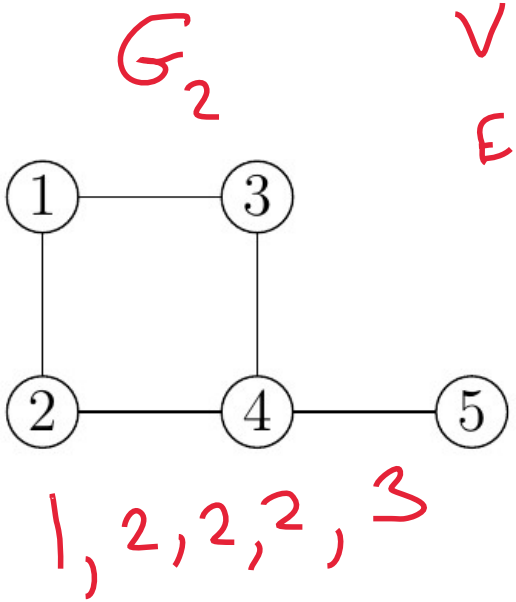
$H = (V', E')$

$V' \subseteq V$

$E' \subseteq E$





Recap: Learning Objectives

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