One-to-One Functions

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- Use WLOG to simplify proofs.

When is a function one-to-one?

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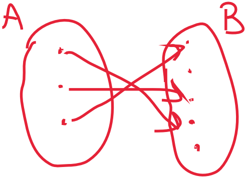
Definition

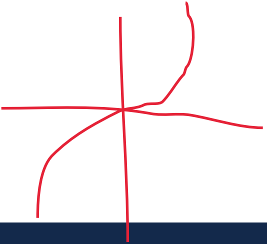
A function is ***one-to-one*** if every element in the co-domain has at most one pre-image. injective

$\forall x,y \in A,\ x \neq y \to f(x) \neq f(y)$

$\forall x,y \in A,\ f(x) = f(y) \to x = y$.



Ex. 1: $f: \mathbb{R} \to \mathbb{R},\ f(x) = x^3$



Ex. 2: $g: \mathbb{N} \to \mathbb{N},\ g(n) = n^2$.

$0 \to 0$

$1 \to 1$

$2 \to 4$

Size Requirements, and Pigeons in Holes

$\underbrace{\text{Students}}_{50} \rightarrow \underbrace{\text{birth months}}_{12}$

Theorem (Pigeonhole Principle)

If n objects are placed into k containers and $n > k$, then at least one container has more than one object.

One container has at least the average. $\frac{n}{k}$

If $n > k \cdot m$, then $\exists$ container w/ at least $m+1$ objects.

$$\frac{n}{k\cancel{m}} > \frac{\cancel{k}m}{\cancel{k}\cancel{m}} = m$$

91 students into 9 Pernsall groups

$\Rightarrow \exists$ group w/ ≥ 11 students.

$|A| > |B| \to f$ not one-to-one

Size Requirements, and Pigeons in Holes

Theorem (Pigeonhole Principle)

If n objects are placed into k containers and $n > k$, then at least one container has more than one object.

Corollary

Given a function $f : A \to B$, if $|A| > |B|$, then f is not one-to-one.

if f one-to-one, then $|A| \le |B|$.

What if a function~~s~~ is both onto and one-to-one?

What if a functions is both onto and one-to-one?

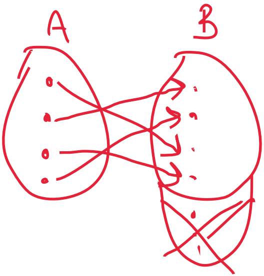
Definition

A function is called a ***bijection*** if it is both onto and one-to-one.

one-to-one requires $|A| \le |B|$

onto requires $|A| \ge |B|$

$\Big\}\ |A| = |B|$



Counting One-to-One Functions

- $f : A \to B$, $|A| = \overset{k}{\cancel{n}}$, $|B| = \overset{n}{\cancel{k}}$

$\cancel{k \cdot k \cdot k \cdots k = k^n}$. n^k.

$$\frac{\left(n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)\right)(n-k)!}{(n-k)!}$$

Counting One-to-One Functions

- $f : A \to B$, $|A| = \overset{k}{\cancel{n}}$, $|B| = \overset{n}{\cancel{k}}$
- $P(n, k) = \frac{n!}{(n-k)!}$

input	output
a_1	
a_2	
$\vdots$	
a_k	

Counting One-to-One Functions

- $f : A \to B$, $|A| = \overset{k}{\cancel{n}}$, $|B| = \overset{n}{\cancel{k}}$
- $P(n, k) = \frac{n!}{(n-k)!} = \frac{n!}{(n-n)!} = n!$
- If $k = n$, $n!$ permutations (one-to-one functions *and* bijections)

Proving one-to-one

Universal claim: $\forall x, y \in A,\ f(x) = f(y) \rightarrow x = y.$

Proving one-to-one

Universal claim:

Example

$f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = 2n - 1$

Proof. Let $x, y \in \mathbb{Z}$ be arb.

Suppose $f(x) = f(y)$.

That is, $2x - 1 = 2y - 1$

$$2x = 2y$$

$$x = y.$$

Hence f is one-to-one. $\square$

Proving not one-to-one

Existential claim: $\neg(\forall x, y \in A,\ x \neq y \rightarrow f(x) \neq f(y))$

$\exists x, y \in A,\ x \neq y$ and $f(x) = f(y)$.

$p \rightarrow q$

$\neg p \vee q$

$p \wedge \neg q$

Proving not one-to-one

Existential claim:

Example

$h : \mathbb{Z} \to \mathbb{Z}$, $h(n) = n^2$

Proof Consider $x = -1$ and $y = 1$.

Then $x \neq y$, but $h(x) = (-1)^2 = 1 = 1^2 = h(y)$.

So h is not one-to-one. $\square$

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- Example: Prove $\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$.

Case 1: $x \geq 0, y \geq 0$

WLOG, suppose $x < 0$.

2: $x < 0, y \geq 0$

3: $x \geq 0, y < 0$

4. $x < 0, y < 0$.

WLOG (No, I didn't misspell vlog)

- WLOG = "Without loss of generality"
- Tool for combining cases in proofs
- Example: Prove $\forall x, y \in \mathbb{R}, |x+y| \leq |x| + |y|$.
- Non-example: Prove that for all $x \in \mathbb{Z}$, $x(x+1)$ is even.

Case 1: x is even.

Case 2: x is odd.

WLOG, suppose x is even.

Recap: Learning Objectives

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