One-to-One Functions

Ian Ludden

Learning Objectives

By the end of this lesson, you will be able to:

Learning Objectives

By the end of this lesson, you will be able to:

- Define and recognize one-to-one functions and bijections.

Learning Objectives

By the end of this lesson, you will be able to:

- Define and recognize one-to-one functions and bijections.
- State and apply the pigeonhole principle.

Learning Objectives

By the end of this lesson, you will be able to:

- Define and recognize one-to-one functions and bijections.
- State and apply the pigeonhole principle.
- Count the number of one-to-one functions from A to B.

Learning Objectives

By the end of this lesson, you will be able to:

- Define and recognize one-to-one functions and bijections.
- State and apply the pigeonhole principle.
- Count the number of one-to-one functions from A to B.
- Prove a given function is (not) one-to-one.

Learning Objectives

By the end of this lesson, you will be able to:

- Define and recognize one-to-one functions and bijections.
- State and apply the pigeonhole principle.
- Count the number of one-to-one functions from A to B.
- Prove a given function is (not) one-to-one.
- Use WLOG to simplify proofs.

When is a function one-to-one?

When is a function one-to-one?

Definition

A function is ***one-to-one*** if every element in the co-domain has at most one pre-image.

Size Requirements, and Pigeons in Holes

Size Requirements, and Pigeons in Holes

Theorem (Pigeonhole Principle)

If n objects are placed into k containers and $n > k$, then at least one container has more than one object.

Size Requirements, and Pigeons in Holes

Theorem (Pigeonhole Principle)

If n objects are placed into k containers and $n > k$, then at least one container has more than one object.

Corollary

Given a function $f : A \to B$, if $|A| > |B|$, then f is not one-to-one.

What if a function is both onto and one-to-one?

What if a function is both onto and one-to-one?

Definition

A function is called a ***bijection*** if it is both onto and one-to-one.

Counting One-to-One Functions

- $f : A \to B$, $|A| = k$, $|B| = n$

Counting One-to-One Functions

- $f : A \to B$, $|A| = k$, $|B| = n$
- $P(n, k) = \frac{n!}{(n-k)!}$

Counting One-to-One Functions

- $f : A \to B$, $|A| = k$, $|B| = n$
- $P(n, k) = \frac{n!}{(n-k)!}$
- If $k = n$, $n!$ permutations (one-to-one functions *and* bijections)

Proving one-to-one

Universal claim:

Proving one-to-one

Universal claim:

Example

$f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = 2n - 1$

Proving not one-to-one

Existential claim:

Proving not one-to-one

Existential claim:

Example

$h : \mathbb{Z} \to \mathbb{Z}$, $h(n) = n^2$

WLOG (No, I didn't misspell vlog)

- WLOG = "Without loss of generality"

WLOG (No, I didn't misspell vlog)

- WLOG = "Without loss of generality"
- Tool for combining cases in proofs

WLOG (No, I didn't misspell vlog)

- WLOG = "Without loss of generality"
- Tool for combining cases in proofs
- Example: Prove $\forall x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.

WLOG (No, I didn't misspell vlog)

- WLOG = "Without loss of generality"
- Tool for combining cases in proofs
- Example: Prove $\forall x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.
- Non-example: Prove that for all $x \in \mathbb{Z}$, $x(x + 1)$ is even.

Recap: Learning Objectives

By the end of this lesson, you will be able to:

- Define and recognize one-to-one functions and bijections.
- State and apply the pigeonhole principle.
- Count the number of one-to-one functions from A to B.
- Prove a given function is (not) one-to-one.
- Use WLOG to simplify proofs.