Proving Properties of Relations

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Learning Objectives

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- Prove properties about relations (formally).

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- Prove a relation is (not) a certain type.

Proving Antisymmetry

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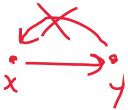
Definition

A relation R on A is ***antisymmetric*** if for all $x, y \in A$ with $x \neq y$, if $x \mathrel{R} y$, then $y \not\mathrel{R} x$.



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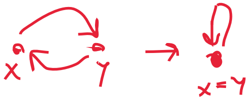

Proving Antisymmetry

$\forall x, y \in A$, if $x R y$ and $y R x$, then $x = y$.

Example

Let $A = \mathbb{R}$, and define a relation R on A as $x \; R \; y$ iff $|x| \leq y$. Prove R is antisymmetric.

Let $x, y \in A$ be arb. and suppose $x R y$ and $y R x$.

That is, $|x| \leq y$ and $|y| \leq x$.

We observe $y \geq 0$ and $x \geq 0$.

So $y = |y|$ and $x = |x|$.

By substitution, $x \leq y$ and $y \leq x$.

Hence $x = y$, and we conclude R is antisymmetric. $\square$

Example

Let $A = \mathbb{R}$, and define a relation R on A as $x\ R\ y$ iff $x \leq |y|$.
Prove/disprove R is a linear order. partial order (RAT)

all pairs comparable (xRy or yRx)

R: Let $x \in A$ be arb.
$x \leq |x|$, so xRx. ✓

A: Let $x, y \in A$ be arb.
Suppose xRy and yRx.
That is, $x \leq |y|$ and $y \leq |x|$.

$\forall x, y \in A,\ xRy \wedge yRx \to x = y$.

Consider $x = -1$ and $y = 1$.
Then $x = -1 \leq |1| = |y|$
and $y = 1 \leq |-1| = |x|$.
So xRy and yRx, but $x \neq y$. Hence R is not antisymm. □

Example

Let $A = \mathbb{Z}^2$, and define a relation ▲ on A as

$\mathbb{Z} \times \mathbb{Z}$

R

$$(a, b) \blacktriangle (x, y) \text{ iff } a = x \text{ and } b = y.$$

RAT

Classify ▲ as ~~a partial order/linear order/strict partial order~~/equivalence relation/none of these, and prove your answer.

RST

(R) Let $(a,b) \in A$ be arb. Then $a=a$ and $b=b$, so $(a,b) \Delta (a,b)$. ✓

(S) Let $(a,b),(x,y) \in A$ be arb. Suppose $(a,b) \neq (x,y)$ and $(a,b) \Delta (x,y)$. This is impossible. By vacuous truth, Δ is symm. ✓

(A)

(T) Let $(a,b), (c,d), (x,y) \in A$. Suppose $(a,b) \Delta (c,d)$ and $(c,d) \Delta (x,y)$. Then $a=c$, $b=d$, $c=x$, and $d=y$. So $a=x$ and $b=y$, and we have $(a,b) \Delta (x,y)$. ✓

Recap: Learning Objectives

By the end of this lesson, you will be able to:

- Prove properties about relations (formally).
- Prove a relation is (not) a certain type.