Special Types of Relations

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Learning Objectives

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- Construct the equivalence classes of an equivalence relation.

Recall: Properties of Relations

- Reflexive/irreflexive/neither

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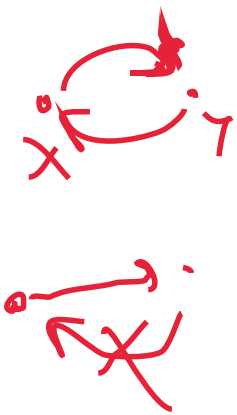
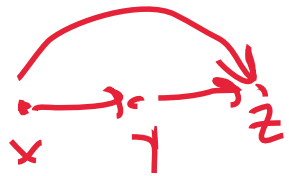
- Reflexive/irreflexive/neither
- Symmetric/antisymmetric/neither



Recall: Properties of Relations

- Reflexive/irreflexive/neither
- Symmetric/antisymmetric/neither
- Transitive/not transitive



Partial Order

Definition

A relation is a ***partial order*** if it is

- reflexive,
- antisymmetric, and
- transitive.

Linear (a.k.a. Total) Order

Definition

A relation is a ***linear order*** if it is

- a partial order, and
- every pair of elements is ***comparable*** ($x \, R \, y$ or $y \, R \, x$).

$\forall x, y \in A$, if $x \neq y$, then $x R y$ $\oplus$ $y R x$.

Strict Partial Order

Definition

A relation is a ***strict partial order*** if it is

- irreflexive,
- antisymmetric, and
- transitive.

$\forall x \in A, \quad x \not R x .$

I
A
T

Equivalence Relation

Definition

A relation is an ***equivalence relation*** if it is

- reflexive, R
- symmetric, and S
- transitive. T

Equivalence Relation

Definition

A relation is an ***equivalence relation*** if it is

- reflexive,
- symmetric, and
- transitive.

Definition

Given an equivalence relation R on a set A, the ***equivalence class*** of x is the set of all elements related to x, denoted

$$[x]_R = \{y \in A : x \, R \, y\} = \{y \in A : y \, R \, x\}.$$

Exercise 1

Classify the relation R on $\mathbb{Z}$ given by $x \, R \, y$ iff $x \equiv y \pmod 7$.

$\mathbb{Z}_7$

$[x]_R = \{y \in \mathbb{Z} : x \equiv y \pmod 7\}$

$[0], [1], \ldots, [6]$

Equivalence relation

R: $x \equiv x$? ✓

S: $x \equiv y \pmod 7 \Rightarrow y \equiv x \pmod 7$? ✓

T: $x \equiv y \pmod 7$ and $y \equiv z \pmod 7$

$\rightarrow x \equiv z \pmod 7$? ✓

Exercise 2

Classify the relation R on $\mathbb{Z}$ given by $x \, R \, y$ iff $x \mid y$.

✓ R: yes, $x \mid x$ $\quad 0 = 0 \cdot 0$

✗ S: $x \mid y \Rightarrow y \mid x$? $3 \mid 6$, but $6 \nmid 3$.

✓ A: $\forall x, y \in \mathbb{Z}$, if $x \neq y$ and $x R y$, then $y \not R x$?

$x \neq y$ and $x \mid y$, then $y = x \cdot$ (integer, not 1)

$y \nmid x$.

✓ T: $x \mid y$ and $y \mid z \rightarrow x \mid z$ ✓

$\Rightarrow$ Partial order. Not a linear order. $17 \nmid 39$

Exercise 3

Classify the relation R on $\mathbb{Z}$ given by $x \, R \, y$ iff $x \mid y$ and $x \neq y$.

R : No

✓ I : Yes

S : No

✓ A : Yes

✓ T : Yes

I
A
T
} strict partial order.

Exercise 4

Classify the relation R on $\mathbb{Z}$ given by $x \, R \, y$ iff $x \leq y$.

R: $x \leq x$? ✓

S: $x \leq y \rightarrow y \leq x$? ✗

A: if $x \neq y$ and $x \leq y$, then $y \not\leq x$? ✓

$y \geq x + 1$

T: $x \leq y$ and $y \leq z \rightarrow x \leq z$? ✓

R, A, T } partial order

linear order

$\forall x, y \in \mathbb{Z}$, if $x \neq y$, is $x \leq y$ or $y \leq x$?

Recap: Learning Objectives

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