# Introduction to Relations

Ian Ludden

# Learning Objectives

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- Define a relation on a set and recall the accompanying notation.
- Represent a relation as a directed graph.
- Restate the formal definitions of standard relation properties and determine which properties a given relation has.

# What is a relation?

## Definition

A ***relation*** $R$ on a (nonempty) set $A$ is a subset of $A \times A$, that is, a set of ordered pairs of elements from $A$. We write $x\ R\ y$ ($x$ relates to $y$) if $(x, y) \in R$ and $x \not R\ y$ ($x$ does not relate to $y$) otherwise.

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## Examples

- $A = \mathbb{Z}$, $x \; R \; y$ iff $|x| = |y|$

$\longleftrightarrow$

$0 \; R \; 0$ $\qquad$ $17 \; R \; 17$

$3 \; R \; -3$ $\qquad$ $31 \not R \; -5$

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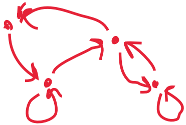
## Examples

- $A = \mathbb{Z}$, $x \, R \, y$ iff $|x| = |y|$ 4 $R$ 8 10 $\not R$ 2
- $A = \mathbb{Z}$, $x \, R \, y$ iff $x \mid y$ −7 $R$ 35 2 $R$ 10

# Relations as Directed Graphs



- Vertex (a.k.a. node) for each element in $A$

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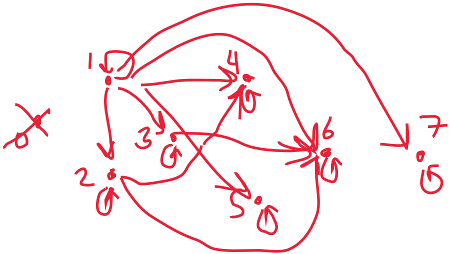
## Examples

- $A = \{a \in \mathbb{N} : 1 \leq a \leq 7\}$, $x \; R \; y$ iff $x \mid y$

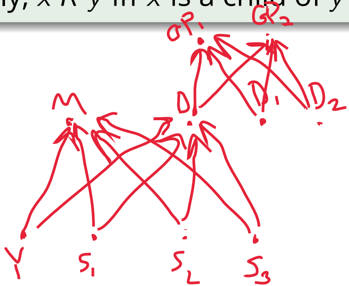


# Relations as Directed Graphs

- Vertex (a.k.a. node) for each element in $A$
- Edge from $x$ to $y$ iff $x \; R \; y$

## Examples

- $A = \{a \in \mathbb{N} : a \leq 7\}$, $x \; R \; y$ iff $x \mid y$
- $A =$ your family, $x \; R \; y$ iff $x$ is a child of $y$



# Properties of Relations: (ir)reflexive

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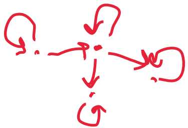
**Definition**

***Reflexive:*** every element relates to itself. $\forall x \in A,\ xRx.$



## Definition

***Reflexive:*** every element relates to itself.

## Definition

***Irreflexive:*** no element relates to itself. $\forall x \in A,\ x \not R x.$

Not reflexive:

$$\neg\left(\forall x \in A,\ xRx\right) = \exists x \in A,\ x \not R x.$$

# Properties of Relations: (ir)reflexive

**Definition**

***Reflexive:*** every element relates to itself.

**Definition**

***Irreflexive:*** no element relates to itself.

**Definition**

Neither: Some elements relate to themselves, but some don't.

# Properties of Relations: (anti)symmetric

$x R y$ iff $|x| = |y|$

# Properties of Relations: (anti)symmetric

**Definition**

***Symmetric:*** all relationships go both directions.

$\forall x, y \in A, x \neq y, xRy \to yRx$

$xRx \to xRx$

# Properties of Relations: (anti)symmetric

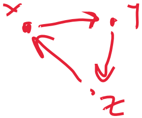
**Definition**

***Symmetric:*** all relationships go both directions.

**Definition**

***Antisymmetric:*** no relationship (between different elements) goes both directions.

$\forall x, y \in A$ with $x \neq y$,
if $xRy$, then $y \not R x$.



**Definition**

***Symmetric:*** all relationships go both directions.

**Definition**

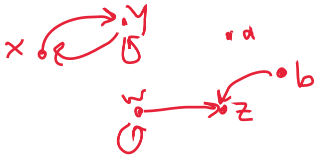
***Antisymmetric:*** no relationship (between different elements) goes both directions.

**Definition**

Neither: Some relationships go both directions, but some don't.

$\exists x, y, w, z \in A,\ x \neq y,\ w \neq z,$

$xRy \wedge yRx \wedge wRz \wedge z \cancel{R} w$



# Properties of Relations: (anti)transitive

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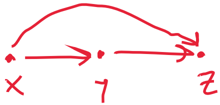
**Definition**

***Transitive:*** For all $x, y, z \in A$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$.



# Properties of Relations: (anti)transitive

## Definition

***Transitive:*** For all $x, y, z \in A$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$.

## Definition

***Antitransitive:*** (optional) For all $x, y, z \in A$, if $x \; R \; y$ and $y \; R \; z$, then $x \not R \; z$.


Not transitive: $\exists x, y, z \in A \quad xRy \wedge yRz \wedge x \not R z$.

$\forall x \in A, x \not R x$.

$R = \emptyset$

$\forall x, y \in A, x \neq y$,
if $xRy$, then $yRx$.

# Properties of Relations: (anti)transitive

**Definition**

***Transitive:*** For all $x, y, z \in A$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$.

**Definition**

***Antitransitive:*** (optional) For all $x, y, z \in A$, if $x \; R \; y$ and $y \; R \; z$, then $x \not R \; z$.

**Definition**

Neither: There are some $x, y, z$ satisfying each conditional statement.

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By the end of this lesson, you will be able to:

- Define a relation on a set and recall the accompanying notation. $(x,y) \in R$ $\quad x R y$
- Represent a relation as a directed graph.
- Restate the formal definitions of standard relation properties and determine which properties a given relation has.